

Higher Dimensional Quantum Hall Effect as A-Class Topological Insulator

Kazuki Hasebe

*Department of Physics, Stanford University, Stanford, California 94305, USA*¹

khasebe@stanford.edu

March 21, 2014

Abstract

We perform a detail study of higher dimensional quantum Hall effects and A-class topological insulators with emphasis on their relations to non-commutative geometry. There are two different formulations of non-commutative geometry for higher dimensional fuzzy spheres; the ordinary commutator formulation and quantum Nambu bracket formulation. Corresponding to these formulations, we introduce two kinds of monopole gauge fields; non-abelian gauge field and antisymmetric tensor gauge field, which respectively realize the non-commutative geometry of fuzzy sphere in the lowest Landau level. We establish connection between the two types of monopole gauge fields through Chern-Simons term, and derive explicit form of tensor monopole gauge fields with higher string-like singularity. The connection between two types of monopole is applied to generalize the concept of flux attachment in quantum Hall effect to A-class topological insulators. We propose tensor type Chern-Simons theory as the effective field theory for membranes in A-class topological insulators. Membranes turn out to be fractionally charged objects and the phase entanglement mediated by tensor gauge field transforms the membrane statistics to be anyonic. The index theorem supports the dimensional hierarchy of A-class topological insulator. Analogies to D-brane physics of string theory are discussed too.

¹On leave from Kagawa National College of Technology, Takuma-cho, Mitoyo, Kagawa 769-1192, Japan. After March 31 2014, email to hasebe@dg.kagawa-nct.ac.jp

Contents

1	Introduction	2
2	Fuzzy Sphere and Dirac Monopole	4
2.1	Hopf map and Lagrange formalism	5
2.2	Hamilton formalism and angular momentum	6
3	Non-commutative Geometry in Higher Dimensions	7
3.1	Fuzzy sphere algebra	7
3.2	Two monopole set-ups for higher dimensional fuzzy sphere	9
4	Non-Abelian Monopole and Higher Dimensional Quantum Hall Effect	10
4.1	$SO(2k)$ non-Abelian monopole	10
4.2	Non-commutative geometry in the lowest Landau level	13
4.3	The $SO(2k + 1)$ Landau model	15
4.4	The $SO(2k + 1)$ spinor Landau model and index theorem	17
4.5	Laughlin-like wavefunction	20
5	Tensor Monopole Fields from Non-Abelian Monopole Fields	21
5.1	Tensor monopole fields	21
5.2	Relation between field strengths of monopoles	22
5.3	Relation between gauge fields of monopoles	25
5.4	Quantum Nambu geometry via tensor monopole	27
6	Flux Attachment and Tensor Chern-Simons Field Theory for Membranes	28
6.1	Basic observations	29
6.2	Tensor flux attachment	31
6.3	$(2k - 2)$ -brane as the $SO(2k + 1)$ skyrmion	34
6.4	Flux cancellation and tensor Chern-Simons theory	35
6.5	Composite membrane and fractional charge	37
6.6	Dimensional hierarchy and analogies to string theory	39
7	Summary and Discussions	41
A	Fully symmetric representations of $SO(2k + 1)$ and $SO(2k)$	42
B	Properties of quantum Nambu bracket	44
C	Winding number for $S^{2k-1} \rightarrow SO(2k)$ and tensor monopole charge	46
D	Linking number between membranes	48

1 Introduction

About a decade ago, the time reversal counterpart of quantum Hall effect, quantum spin Hall effect, was theoretically proposed and experimentally discovered [1, 2, 3, 4]. Since then, topological states of matter have been vigorously investigated [See reviews as Refs.[5, 6, 7]]. Now, we understand there exist a variety of topological cousins of quantum Hall effect, such as topological insulators with time reversal symmetry and topological superconductors with particle hole symmetry. Based on a generalized Altland and Zirnbauer random matrix, a systematic classification of the band topological insulators was exhausted in the periodic table of ten-fold way [8, 9, 10, 11], where we readily find topological insulators in any dimension with or without three discrete symmetries, time reversal, particle-hole, and chiral. For instance, the quantum Hall effect is assigned to the lowest dimensional (2D) entity of the A-class topological insulators that do not respect any of the three discrete symmetries and live in arbitrary even dimensional space. A-class topological insulators are regarded as a higher dimensional counterpart of the quantum Hall effect.

Recently, theoretical realizations of fractional version of topological insulators have been proposed [12, 13], and two groups independently applied the non-commutative geometry techniques to fractional topological insulators [14, 15] generalizing the techniques used in 2D quantum Hall effect [16, 17, 18, 19]. In the works, they proposed quantum Nambu geometry [20, 21] as underlying mathematics of topological insulators. In particular, close relations between quantum Nambu bracket in even dimensions and A-class topological insulator were pointed out in Ref.[14] where monopole in the momentum space generates the non-commutativity of density operators. Since A-class topological insulators are a natural higher dimensional counterpart of quantum Hall effect, A-class topological insulators give a good starting point to see how non-commutative geometry works in topological insulators before discussing more “complicated” topological insulators, such as AII class². Before the discovery of topological insulators, 4D generalization of quantum Hall effect was theoretically proposed in the $SU(2)$ monopole background by Zhang and Hu [27] as a generalization of the Haldane’s quantum Hall effect on two-sphere [28]. In general, higher dimensional quantum Hall effects are realized in (color) monopole gauge field background compatible with the holonomy group of the basemanifold on which the system is defined [29, 30, 31]. Since there exists magnetic field of monopole, higher dimensional quantum Hall effects necessarily break time-reversal symmetry as A-class topological insulators are ought to do. The higher dimensional quantum Hall effect can be considered as a realization of A-class topological insulators with Landau level³. From this perspective, we revisit the higher dimensional quantum Hall effect that are realized on arbitrary even-dimensional spheres [31, 32]. In the set-up of quantum Hall effect on S^{2k} , the $SO(2k)$ non-abelian monopole is adopted, and the system realizes interesting mathematical structures. For instance, the non-abelian monopole mathematically corresponds to the sphere-bundle over sphere [33] where the S^{2k-1} -bundle over the base manifold S^{2k} gives the $SO(2k)$ structure group. In non-commutative geometry point of view, the system can be regarded as a physical set-up of higher dimensional fuzzy sphere in the lowest Landau level⁴. Interestingly,

²Recently, AII topological insulators with Landau level were constructed in Refs.[25, 26].

³In this sense, the 4D quantum Hall effect was the firstly “discovered” higher dimensional topological insulator.

⁴Such physical description of fuzzy sphere in monopole background is “consistent” with the dielectric effect of

higher dimensional quantum Hall effects are even related to supersymmetry [36, 37] and twistor theory [29, 38, 39].

Though in the former articles, the non-abelian monopoles are adopted, there may be another monopole realization for higher dimensional quantum Hall effect. That is to use antisymmetric *tensor* $U(1)$ monopole. Tensor $U(1)$ monopole is a monopole [40, 41] whose gauge group is $U(1)$ but gauge field is *not* a vector but an antisymmetric tensor⁵. While the non-abelian monopole corresponds to an extension of the Dirac monopole by increasing the *internal* gauge degrees of freedom, the tensor monopole manifests another extension of the Dirac monopole by increasing the *external* indices. Therefore, there may be two reasonable generalizations of quantum Hall effect, one is based on the non-abelian monopole and the other is based on the tensor monopole. One may be immediately inclined to ask the following questions. What does quantum Hall effect in tensor monopole background look like and what kind of non-commutative geometry will emerge in the lowest Landau level? If 2D quantum Hall effect has two reasonable generalizations, is there any connection between them? For such questions, researches of non-commutative geometry gives a suggestive hint; There are two (superficially) different formulations for higher dimensional generalizations fuzzy sphere [22, 23, 24], one of which is the ordinary commutator formulation and the other is the quantum Nambu bracket formulation. Inspired by the observation, we establish connection between the non-abelian and tensor monopole and answer to the questions in this work.

Topological field theory description of the quantum Hall effect [43, 44] has brought great progress in understanding non-perturbative aspects of quantum Hall effect. The Chern-Simons effective field theory naturally describes the flux attachment that electron and Chern-Simons fluxes are combined to yield a “new particle” called composite boson [45, 46], and the fractional quantum Hall effect is regarded as a superfluid state of the composite bosons [44]. The fundamental object of the A-class topological insulator turns out to be membrane-like objects. Based on the connection between the non-abelian and tensor monopoles, we propose a tensor type Chern-Simons field theory as the effective field theory of the A-class topological insulator. Interestingly, while we start from the non-abelian quantum mechanics in $(2k + 1)$ D space-time, the tensor Chern-Simons field theory is defined in $(4k - 1)$ D space-time. Membranes have a fractional charge and obey anyonic statistics. Ground state of A-class topological insulators is regarded as a superfluid state of composite membrane at magic values of the filling factor. We discuss dimensional condensation of membranes with emphasis on relation to brane-democracy of string theory. We thus integrate so far loosely connected subjects, such as Nambu-bracket, tensor topological field theory and physics of quantum Hall effect, to have an entire picture of A-class topological insulator [Fig.1].

Though we share several terminologies with string theory such as p -branes and C field, the present analysis is not directly related to the string theory: We do not use either strings or D-branes. About a realization of topological insulators in string theory, one may consult [47, 48]. For C field realization of non-commutative geometry on M-brane, see Refs.[49, 50, 51].

The paper is organized as follows. In Sec.2, we briefly review the basic mathematics of the fuzzy sphere and its physical realization in the lowest Landau level. Sec.3 describes the two mathematical

D-brane [34, 35].

⁵Such antisymmetric tensor gauge field is also known as Kalb-Ramond field [42].

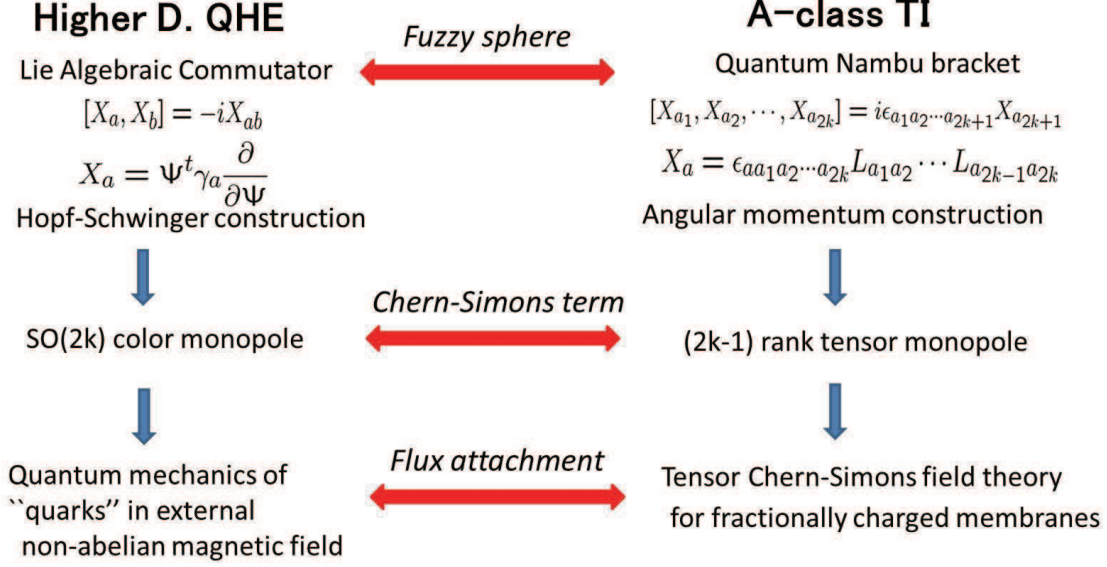


Figure 1: Correspondence between mathematics and physics of higher dimensional quantum Hall effects and A-class topological insulators.

formulations for higher dimensional fuzzy spheres. We introduce non-abelian monopole quantum Hall effect with or without spin degrees of freedom in Sec.4. Sec.5 discusses the connection between the tensor and non-abelian monopoles, and gives a tensor monopole realization of the quantum Nambu geometry. In Sec.6, the Chern-Simons tensor field theory is proposed as the effective field theory of A-class topological insulator, where we clarify the fractional charge and anyonic statistics of membranes. We also discuss the hierarchical property of membranes and A-class topological insulator. Sec.7 is devoted to summary and discussions.

2 Fuzzy Sphere and Dirac Monopole

Here, we briefly review how the fuzzy geometry emerges in the context of the lowest Landau level physics by using the fuzzy two-sphere and Dirac monopole system. The observation is a template for higher dimensional fuzzy sphere in the subsequent sections.

The fuzzy two-sphere [52, 53, 54] is a fuzzy manifold whose coordinates X_i ($i = 1, 2, 3$) satisfy the $SU(2)$ algebra:

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k, \quad (1)$$

and

$$X_i X_i = \left(\frac{\alpha}{2}\right)^2 I(I+2) = r^2(1 + \frac{2}{I}). \quad (2)$$

Here, α is the unit of non-commutative length and I (integer) specifies the radius of the fuzzy two-sphere r as

$$r = \frac{\alpha}{2}I. \quad (3)$$

The fuzzy sphere is realized as the lowest Landau level physics. We will show how fuzzy geometry emerges on a two-sphere in Dirac monopole background, respectively from the Lagrange and Hamilton formalisms .

2.1 Hopf map and Lagrange formalism

The Lagrangian for the electron on a two-sphere in monopole background is given by

$$L = \frac{M}{2} \dot{x}_i \dot{x}_i - \dot{x}_i A_i, \quad (4)$$

where x_i ($i = 1, 2, 3$) are subject to a constraint

$$x_i x_i = r^2, \quad (5)$$

and the Dirac monopole gauge field is given by

$$A_i = -\frac{I}{2r(r+x_3)} \epsilon_{ij3} x_j, \quad (6)$$

with Dirac monopole charge $I/2$ (I integer) [61]. Relation to the non-commutative geometry will be transparent by introducing the Hopf spinor. The Hopf spinor is the two-component spinor that induces the (1st) Hopf map $S^3 \xrightarrow{S^1} S^2$:

$$\phi \rightarrow x_i = \frac{\alpha}{2} \phi^\dagger \sigma_i \phi, \quad (7)$$

with

$$\phi^\dagger \phi = I. \quad (8)$$

The x_a (7) given by automatically satisfy the condition of two-sphere:

$$x_i x_i = \left(\frac{\alpha}{2}\right)^2 (\phi^\dagger \phi)^2 = r^2. \quad (9)$$

The Hopf spinor ϕ takes the form

$$\phi = \sqrt{\frac{I}{2r(r+x_3)}} \begin{pmatrix} r+x_3 \\ x_1+ix_2 \end{pmatrix} e^{i\chi} \quad (10)$$

with $e^{i\chi}$ denoting $U(1)$ phase factor, and the monopole gauge field (6) can be derived as

$$A = A_i dx_i = -i\phi^\dagger d\phi. \quad (11)$$

In the lowest Landau level, the kinetic energy is quenched and the Lagrangian (4) is reduced to the following form

$$L_{\text{LLL}} = -A_i \dot{x}_i = i\phi^\dagger \frac{d}{dt} \phi. \quad (12)$$

We regard the Hopf spinor as the fundamental variable and derive the canonical momentum of ϕ as $i\phi^*$ from (12), and apply the quantization condition to them (not the original coordinates x_i):

$$[\phi_\alpha, \phi_\beta^*] = \delta_{\alpha\beta}. \quad (13)$$

After the quantization, the Hopf spinor becomes to the Schwinger operator of harmonic oscillator expressed as⁶:

$$\phi_\alpha, \phi_\beta^* \rightarrow \frac{\partial}{\partial \phi_\alpha}, \phi_\beta, \quad (14)$$

and the coordinates on two-sphere (7) turn out to be

$$X_i = \frac{\alpha}{2} \phi^t \sigma_i \frac{\partial}{\partial \phi}, \quad (15)$$

which satisfy the fuzzy two-sphere algebra (1), and the condition (8) is rewritten as

$$\phi^t \frac{\partial}{\partial \phi} = I. \quad (16)$$

One can readily show that Eq.(15) with(16) indeed satisfies (2). The emergence of fuzzy sphere is based on the Hopf-Schwinger operator and the Pauli matrices in the Lagrange formalism.

2.2 Hamilton formalism and angular momentum

The Hamiltonian for a particle in gauge field is given by

$$H = -\frac{1}{2M} D_i^2 = -\frac{1}{2M} \frac{\partial^2}{\partial r^2} - \frac{1}{Mr} \frac{\partial}{\partial r} + \frac{1}{2Mr^2} \Lambda_i^2, \quad (17)$$

where D_i represent the covariant derivative:

$$D_i = \partial_i + iA_i, \quad (18)$$

and Λ_i denote the covariant angular momentum:

$$\Lambda_i = -i\epsilon_{ijk} x_j D_k. \quad (19)$$

Hence, the Hamiltonian for a particle on two-sphere (r const.) is given by

$$H = \frac{1}{2Mr^2} \Lambda_i^2. \quad (20)$$

With the $U(1)$ monopole at the center of the sphere, the total angular momentum L_i is given by the sum of the covariant angular momentum and the angular momentum of monopole gauge field:

$$L_i = \Lambda_i + r^2 F_i = \Lambda_i + \frac{1}{\alpha} x_i, \quad (21)$$

where

$$F_i = \epsilon_{ijk} \partial_j A_k = \frac{I}{2r^3} x_i. \quad (22)$$

Since L_i are the conserved angular momentum, they satisfy the $SU(2)$ algebra

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \quad (23)$$

⁶We can derive the same result in the Hamilton formalism. The lowest Landau level eigenstates are given by the holomorphic function of ϕ , and the complex conjugate of ϕ is effectively represented by the derivative of ϕ .

In the lowest Landau level, the kinetic term is quenched $\Lambda_i = 0$, and then x_i ($\propto F_i$) can be identified with L_i :

$$X_i = \alpha L_i. \quad (24)$$

It is obvious that X_i satisfy the fuzzy two-sphere algebra (1). With use of $L_{ij} = \epsilon_{ijk} L_k$, (24) is written as

$$X_i = \frac{\alpha}{2} \epsilon_{ijk} L_{jk}. \quad (25)$$

Notice the construction of fuzzy sphere coordinates in the Hamilton formalism is based on the angular momentum.

Consequently, there are two ways to see the emergence of fuzzy sphere, one of which is the Hopf-Schwinger construction (15) in the Lagrange formalism, and the other is the angular momentum construction (25) in the Hamilton formalism.

3 Non-commutative Geometry in Higher Dimensions

3.1 Fuzzy sphere algebra

As discussed above the coordinates of fuzzy two-sphere are given by the $SO(3)$ vector operators that satisfy

$$[X_i, X_j] = i\alpha \epsilon_{ijk} X_k,$$

and its minimal representation is given by the 2×2 Pauli matrices. Since Pauli matrices are equal to the $SO(3)$ gamma matrices, it may be natural to adopt the $SO(2k+1)$ gamma matrices as the coordinates of S_F^{2k} with minimum radius. For S_F^{2k} with larger radius, the $SO(2k+1)$ gamma

matrices G_a ($a = 1, 2, \dots, 2k+1$) of fully symmetric representation⁷, $\overbrace{[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}]}^k$, is adopted as the fuzzy coordinates[55, 56]. Indeed $X_a \equiv \alpha G_a$ satisfy

$$\sum_{a=1}^{2k+1} X_a X_a = \frac{\alpha^2}{4} I(I+2k) = r^2(1 + \frac{2k}{I}), \quad (26)$$

which represents the condition of constant radius of the fuzzy sphere. In the limit $I \rightarrow \infty$ with fixed r , (26) is reduced to the condition of the classical $2k$ -sphere, $\sum_{a=1}^{2k+1} x_a x_a = r^2$.

One should notice however, there is a big difference between the fuzzy two-sphere and its higher dimensional counterpart [57, 58, 59, 60]. Though the $SO(3)$ gamma matrices are equivalent to the $SO(3)$ generators and form a closed algebra by themselves, the $SO(2k+1)$ ($k \geq 2$) gamma matrices X_a do not satisfy a closed algebra among them but their commutators yield “new” operators, the $SO(2k+1)$ generators X_{ab} :

$$[X_a, X_b] = i\alpha X_{ab}. \quad (27)$$

⁷For several properties of gamma matrix in fuzzy symmetric representation, see Append.A.

The appearance of X_{ab} suggests that the geometry of higher dimensional fuzzy sphere cannot simply be understood only by the original coordinates. To construct a closed algebra for higher dimensional fuzzy sphere, we need to incorporate X_{ab} also to have an enlarged algebra

$$\begin{aligned}[X_a, X_{bc}] &= -i\alpha(\delta_{ab}X_c - \delta_{ac}X_b), \\ [X_{ab}, X_{cd}] &= i\alpha(\delta_{ac}X_{bd} - \delta_{ad}X_{bc} + \delta_{bd}X_{ac} - \delta_{bc}X_{ad}).\end{aligned}\quad (28)$$

in which X_a and X_{ab} amount to the $SO(2k+2)$ algebra. Around the north pole, (27) reduces to

$$[X_\mu, X_\nu] = i\alpha\eta_{\mu\nu}^i X_i, \quad (29)$$

where $\eta_{\mu\nu}^i$ denotes the expansion coefficient (for $k=2$, $\eta_{\mu\nu}^i$ is given by the t'Hooft symbol) and X_i stand for $SO(2k)$ generators related to $X_{\mu\nu}$ by the relation

$$X_{\mu\nu} = \sum_{i=1}^{k(2k-1)} \eta_{\mu\nu}^i X_i. \quad (30)$$

The extra-degrees of freedom is described by the operators X_i , and can be interpreted as the fuzzy fibre over S^{2k} . Since the corresponding algebra of S_F^{2k} is the $SO(2k+2)$ algebra, the fuzzy fibre described by the $SO(2k)$ algebra is identified with S_F^{2k-2} . Due to the existence of the fuzzy bundle, the classical counterpart of S_F^{2k} does not simply realize $S^{2k} \simeq SO(2k+1)/SO(2k)$ but $SO(2k)/U(k)$ fibration over S^{2k} [58]

$$S_F^{2k} \simeq SO(2k+1)/U(k) \sim S^{2k} \otimes SO(2k)/U(k). \quad (31)$$

Here, \sim denotes the local equivalence. The $SO(2k)/U(k)$ -fibre is the classical counterpart of the extra fuzzy space S_F^{2k-2} . As we shall see later, such extra degrees of freedom correspond to (fuzzy) membrane excitation.

Though in the commutator formulation, the existence of the fuzzy fibre is explicit, the commutator formulation is rather “awkward” in the sense the algebra does not close within the original fuzzy coordinates. The Nambu bracket gives a more sophisticated formulation. In the d dimension, quantum Nambu bracket (or Nambu-Heisenberg bracket) [21, 22, 23, 24] is defined as

$$[X_{a_1}, X_{a_2}, \dots, X_{a_n}] \equiv X_{[a_1} X_{a_2} \dots X_{a_n]}, \quad (32)$$

where $a_1, a_2, \dots, a_n = 1, 2, \dots, d$ (with $n \leq d$)⁸, and the bracket for the low indices represents the fully anti-symmetric combination about the indices. We have $n!$ terms on the right-hand side of (32). For instance,

$$\begin{aligned}[X_{a_1} X_{a_2}] &= X_{a_1} X_{a_2} - X_{a_2} X_{a_1}, \\ [X_{a_1} X_{a_2} X_{a_3}] &= X_{a_1} X_{a_2} X_{a_3} - X_{a_1} X_{a_3} X_{a_2} + X_{a_2} X_{a_3} X_{a_1} - X_{a_2} X_{a_1} X_{a_3} + X_{a_3} X_{a_1} X_{a_2} - X_{a_3} X_{a_2} X_{a_1}\end{aligned}$$

In the quantum Nambu bracket formulation⁹, the non-commutative algebra for S_F^{2k} is given by [22, 23, 24]

$$[X_{a_1}, X_{a_2}, X_{a_3}, \dots, X_{a_{2k}}] = i^k C(k, I) \alpha^{2k-1} \epsilon_{a_1 a_2 a_3 \dots a_{2k+1}} X_{a_{2k+1}}, \quad (33)$$

⁸For $n > d$, due to the anti-symmetric property, quantum Nambu bracket always vanishes

⁹(33) essentially comes from the following property of the $SO(2k+1)$ gamma matrices, $\gamma_1 \gamma_2 \gamma_3 \dots \gamma_{2k} = i^k \gamma_{2k+1}$. For more detail properties of quantum Nambu bracket, see Appendix B.

where

$$C(k, I) = \frac{(2k)!!(I + 2k - 2)!!}{2^{2k-1} I!!}. \quad (34)$$

Thus, the extra operators X_{ab} do not appear in the quantum Nambu bracket formulation of fuzzy sphere, and the closure of algebra is satisfied only by the original fuzzy coordinates. The extra fuzzy-fibre degrees of freedom seems to be completely “hidden” in the quantum Nambu bracket. Around the north-pole $X_{2k+1} \simeq r$, (33) is reduced to the quantum Nambu bracket for non-commutative plane:

$$[X_{\mu_1}, X_{\mu_2}, X_{\mu_3}, \dots, X_{\mu_{2k}}] = i^k \ell^{2k} \epsilon_{\mu_1 \mu_2 \mu_3 \dots \mu_{2k}}, \quad (35)$$

where

$$\ell \equiv \alpha \left(\frac{I}{2} C(k, I) \right)^{\frac{1}{2k}} = r \left(\frac{(2k)!!(I + 2k - 2)!!}{I!! I^{2k-1}} \right)^{\frac{1}{2k}} \underset{I \rightarrow \infty}{\sim} \frac{r}{\sqrt{I}}. \quad (36)$$

For instance,

$$\begin{aligned} k=1 & : \quad \ell = r \left(\frac{2}{I} \right)^{\frac{1}{2}}, \\ k=2 & : \quad \ell = r \left(\frac{8(I+2)}{I^3} \right)^{\frac{1}{4}}, \\ k=3 & : \quad \ell = r \left(\frac{48(I+2)(I+4)}{I^5} \right)^{\frac{1}{6}}. \end{aligned} \quad (37)$$

3.2 Two monopole set-ups for higher dimensional fuzzy sphere

As discussed in Sec.2, the fuzzy two-sphere is realized in the Dirac monopole background. The easiest way to find what kind of monopole corresponds to non-commutative geometry is to find the right-hand side of the non-commutative algebra. For instance, the fuzzy two-sphere algebra is given by

$$[X_i, X_j] = i\alpha \epsilon_{ijk} X_k, \quad (38)$$

and one can read off the $U(1)$ monopole field strength from the right-hand side:

$$F_{ij} \simeq \frac{1}{r^3} \epsilon_{ijk} x_k. \quad (39)$$

For higher dimensional fuzzy sphere, in correspondence to the two non-commutative formulations, we will obtain two different types of monopoles.

- Non-abelian monopole

Around the north pole, the commutation relation between the fuzzy coordinates (27) becomes to

$$[X_\mu, X_\nu] = i\alpha X_{\mu\nu},$$

where the right-hand side is the $SO(2k)$ generators. This suggests the field strength of the non-abelian monopole

$$F_{\mu\nu} \simeq \frac{1}{r^2} \Sigma_{\mu\nu}, \quad (40)$$

where $\Sigma_{\mu\nu}$ denotes the $SO(2k)$ matrix generators. Thus we may identify the one monopole set-up for S_F^{2k} with the $SO(2k)$ non-abelian monopole.

- Tensor monopole set-up

Meanwhile, the right-hand side of the quantum Nambu bracket formulation

$$[X_{a_1}, X_{a_2}, \dots, X_{a_{2k}}] = i^k C(k, I) \alpha^{2k-1} \epsilon_{a_1 a_2 \dots a_{2k} a_{2k+1}} X_{a_{2k+1}},$$

implies antisymmetric tensor monopole field strength:

$$G_{a_1 a_2 \dots a_{2k}} \simeq \frac{1}{r^{2k+1}} \epsilon_{a_1 a_2 \dots a_{2k} a_{2k+1}} x_{a_{2k+1}}. \quad (41)$$

Here two comments are added. Firstly, even though there are two different non-commutative formulations, they describe the same non-commutative object, *i.e.* the fuzzy sphere. Similarly, the two different types of monopoles are expected to describe same physical system corresponding to fuzzy sphere. In other words, the non-abelian and the tensor monopoles are two different physical set-ups for the same system. Hence, they are expected to be “equal” in some sense. Their connection will be clarified in Sec.5. Secondly, though the quantum Nambu bracket veils the “extra” degrees of freedom of fuzzy-bundle, $(2k-1)$ rank field (41) on the right-hand side of (35) implies the existence of $(2k-2)$ -brane whose $(2k-1)$ -form current naturally coupled to $(2k-1)$ rank tensor field. This observation will be important in constructing the Chern-Simons tensor field theory in Sec.6.

4 Non-Abelian Monopole and Higher Dimensional Quantum Hall Effect

Here, we give non-abelian monopole realization for higher dimensional quantum Hall effect [31, 32]. The monopole gauge group is chosen $SO(2k)$ so as to be compatible with the holonomy of the basemanifold S^{2k} ¹⁰.

4.1 $SO(2k)$ non-Abelian monopole

First let us introduce the generalized Hopf map:

$$x_a = \alpha \Psi^\dagger \Gamma_a \Psi, \quad (42)$$

¹⁰ The present monopole set-up is quite similar to the Kaluza-Klein monopole in the sense that the geometrical information determines the corresponding monopole gauge group. Kaluza-Klein monopole accompanies with the spontaneous compactification of the Kaluza-Klein theory [62, 63], and the isometry of the compactified space is transferred to the gauge symmetry of the uncompactified space. For instance, S^{2k-1} compactified space yields the gauge field of $SO(2k)$ non-Abelian monopole [64].

where x_a ($a = 1, 2, \dots, 2k+1$) are subject to the condition of S^{2k} :

$$x_a x_a = r^2, \quad (43)$$

and Γ_a ($a = 1, 2, \dots, 2k+1$) denote the $SO(2k+1)$ gamma matrices:

$$\Gamma_i = \begin{pmatrix} 0 & i\gamma_i \\ -i\gamma_i & 0 \end{pmatrix}, \quad \Gamma_{2k} = \begin{pmatrix} 0 & \mathbf{1}_{2^{k-1}} \\ \mathbf{1}_{2^{k-1}} & 0 \end{pmatrix}, \quad \Gamma_{2k+1} = \begin{pmatrix} \mathbf{1}_{2^{k-1}} & 0 \\ 0 & -\mathbf{1}_{2^{k-1}} \end{pmatrix}, \quad (44)$$

with $SO(2k-1)$ gamma matrices γ_i ($i = 1, 2, \dots, 2k-1$). The $SO(2k)$ generators

$$\Sigma_{\mu\nu} \equiv -i\frac{1}{4}[\Gamma_\mu, \Gamma_\nu]. \quad (45)$$

take the form of

$$\Sigma_{\mu\nu} = \begin{pmatrix} \Sigma_{\mu\nu}^+ & 0 \\ 0 & \Sigma_{\mu\nu}^- \end{pmatrix}, \quad (46)$$

where the $SO(2k)$ Weyl generators are

$$\Sigma_{\mu\nu}^\pm = \{\Sigma_{ij}^\pm, \Sigma_{i,2k}^\pm\} = \{-i\frac{1}{2}\gamma_i\gamma_j, \pm\frac{1}{2}\gamma_i\}. \quad (i \neq j) \quad (47)$$

Notice that the $SO(2k)$ Weyl generators (47) consist of the $SO(2k-1)$ generators and the $SO(2k-1)$ gamma matrices. The 2^k component spinor Ψ that satisfies (42) is given by

$$\Psi = \frac{1}{\sqrt{2r(r+x_{2k+1})}} \begin{pmatrix} (r+x_{2k+1})\mathbf{1}_{2^{k-1}} \\ x_{2k}\mathbf{1}_{2^{k-1}} - ix_i\gamma_i \end{pmatrix} \psi, \quad (48)$$

where ψ is a 2^{k-1} component normalized complex spinor $\psi^\dagger\psi = I$. With use of Ψ , the $SO(2k)$ non-abelian gauge fields [65, 66, 67, 68, 69] can be derived by the formula

$$A = -i\Psi^\dagger d\Psi, \quad (49)$$

where $A = A_a dx_a$ with

$$A_\mu = -\frac{1}{r(r+x_{2k+1})} \Sigma_{\mu\nu}^+ x_\nu, \quad (\mu, \nu = 1, 2, \dots, 2k), \\ A_{2k+1} = 0. \quad (50)$$

The field strength $F = dA + iA^2$ or $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ (for $F = \frac{1}{2}F_{ab}dx_a \wedge dx_b$) is evaluated as ¹¹

$$\begin{aligned} F_{\mu\nu} &= -\frac{1}{r^2}x_\mu A_\nu + \frac{1}{r^2}x_\nu A_\mu + \frac{1}{r^2}\Sigma_{\mu\nu}^+, \\ F_{\mu,2k+1} &= \frac{1}{r^2}(r + x_{2k+1})A_\mu. \end{aligned} \quad (54)$$

Around the north pole, $x_{2k+1}/r \simeq 1$, $x_\mu/r \simeq 0$, the field strength (54) is reduced to (40). It is obvious that under the $SO(2k)$ gauge transformation

$$\Psi \rightarrow \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \Psi, \quad (55)$$

with g

$$g = \frac{1}{\sqrt{1 - x_{2k+1}^2}}(x_{2k}\mathbf{1}_{2k-1} + ix_i\gamma_i), \quad (56)$$

A and F are transformed as

$$\begin{aligned} A &\rightarrow g^\dagger A g - ig^\dagger dg, \\ F &\rightarrow g^\dagger F g. \end{aligned} \quad (57)$$

The homotopy theorem guarantees the non-trivial bundle topology¹² of the $SO(2k)$ monopole on S^{2k} :

$$\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z}, \quad (58)$$

which is measured by the k th Chern-number:

$$c_k = \frac{1}{k!(2\pi)^k} \int_{S^{2k}} \text{tr} F^k. \quad (59)$$

¹¹ The component fields of A_a and F_{ab} are respectively given by

$$A_a = \sum_{\mu < \nu} A_a^{\mu\nu} \Sigma_{\mu\nu}^+, \quad F_{ab} = \sum_{\mu < \nu} F_{ab}^{\mu\nu} \Sigma_{\mu\nu}^+, \quad (51)$$

where

$$A_a^{\mu\nu} = -\frac{1}{r(r + x_{2k+1})}(\delta_{a\mu}x_\nu - \delta_{a\nu}x_\mu). \quad (52)$$

and

$$\begin{aligned} F_{\rho\sigma}^{\mu\nu} &= \frac{1}{r^3(r + x_{2k+1})}(\delta_{\rho\mu}x_\sigma x_\nu - \delta_{\rho\nu}x_\sigma x_\mu + \delta_{b\mu}x_\rho x_\nu - \delta_{\sigma\nu}x_\rho x_\mu) + \frac{1}{r^2}(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}), \\ F_{\rho,2k+1}^{\mu\nu} &= -\frac{1}{r^3}(\delta_{\rho\mu}x_\nu - \delta_{\rho\nu}x_\mu). \end{aligned} \quad (53)$$

¹²For $k = 2, 4$ we have two \mathbb{Z} s: $\pi_3(SO(4)) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $\pi_7(SO(8)) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

In low dimensions, (59) yields

$$\begin{aligned}
c_{k=1} &= \frac{1}{2\pi} \int_{S^2} \text{tr} F, \\
c_{k=2} &= \frac{1}{8\pi^2} \int_{S^4} \text{tr} F^2, \\
c_{k=3} &= \frac{1}{48\pi^3} \int_{S^6} \text{tr} F^3, \\
c_{k=4} &= \frac{1}{384\pi^4} \int_{S^8} \text{tr} F^4.
\end{aligned} \tag{60}$$

For the $SO(2k)$ fully symmetric representation $\overbrace{[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}]}^k$, the Chern-numbers are calculated as [35]

$$\begin{aligned}
c_{k=1} &= I, \\
c_{k=2} &= \frac{1}{6} I(I+1)(I+2), \\
c_{k=3} &= \frac{1}{360} I(I+1)(I+2)^2(I+3)(I+4), \\
c_{k=4} &= \frac{1}{302400} I(I+1)(I+2)^2(I+3)^2(I+4)^2(I+5)(I+6),
\end{aligned} \tag{61}$$

which correspond to the monopole charge or the number of magnetic fluxes on spheres.

4.2 Non-commutative geometry in the lowest Landau level

Following to the similar step in Sec.2.1, we can find how higher dimensional fuzzy sphere geometry emerges in the lowest Landau level. It should be noted since the monopole gauge field is non-abelian, and then the particle on S^{2l} carries the $SO(2k)$ color degrees of freedom like a “quark”. The Lagrangian is given by

$$L = \frac{M}{2} \dot{x}_a \dot{x}_a - \dot{x}_a A_a, \tag{62}$$

where $x_a x_a = r^2$. In the lowest Landau level, the Lagrangian is reduced to

$$L = i\Psi^\dagger \frac{d}{dt} \Psi, \tag{63}$$

with Ψ (48). By imposing the canonical quantization condition on Ψ and Ψ^* , x_a (42) are realized as the operators

$$X_a = \frac{\alpha}{2} \Psi^t \Gamma_a \frac{\partial}{\partial \Psi}, \tag{64}$$

which satisfy

$$[X_a, X_b] = i\alpha X_{ab}, \tag{65}$$

where

$$X_{ab} = \alpha \Psi^t \Sigma_{ab} \frac{\partial}{\partial \Psi}, \tag{66}$$

with $\Sigma_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b]$. X_a and X_{ab} amount to $(2k+1) + k(2k+1) = (k+1)(2k+1)$ generators of the $SO(2k+2)$ algebra, and X_{ab} bring the degrees of freedom of fuzzy fibre S_F^{2k-2} over S^{2k} . It should be noted that the coordinates of the external space and those of the internal space are related by (65) and they are same size matrices of the $SO(2k+2)$ generators. Since they are similarly treated in the fuzzy algebra, there is no reason to distinguish the external and internal spaces in the lowest Landau level. Inversely it may be natural to consider an enlarged space that includes both external and internal spaces. Since the fuzzy-fibre coordinates X_{ab} are the $SO(2k+1)$ generators, X_{ab} can be represented as

$$X_{ab} = \alpha L_{ab}. \quad (67)$$

Meanwhile we have $L_{ab} \sim r^2 F_{ab}$ in the lowest Landau level (see Sec.5.4). From these relations, we have

$$X_{ab} \sim \alpha r^2 F_{ab}, \quad (68)$$

which suggests the non-abelian field strength is equivalent to the fuzzy-fibre [see Fig.2]. This identification coincides with the intuitive picture, since the fuzzy-fibre realizes the non-abelian flux of the monopole. In the 2D quantum Hall liquid, the $U(1)$ magnetic flux penetration induces a charged excitation at the point where the flux is pierced. Similarly in higher dimensional quantum Hall liquid, the non-abelian flux penetration will induce an point-like excitation on S^{2k} . Though the excitation is a “point”-like object on S^{2k} , the non-abelian flux accommodates the S_F^{2k-2} geometry as its internal space. Remember that there is no distinction between the external and internal spaces in the lowest Landau level, and so the “internal” space S_F^{2k-2} can be regarded as an extended $(2k-2)$ dimensional object, $(2k-2)$ -brane, in the enlarged $(4k-2)$ dimensional space. In this sense, the non-abelian flux penetration induces $(2k-2)$ -brane excitation.

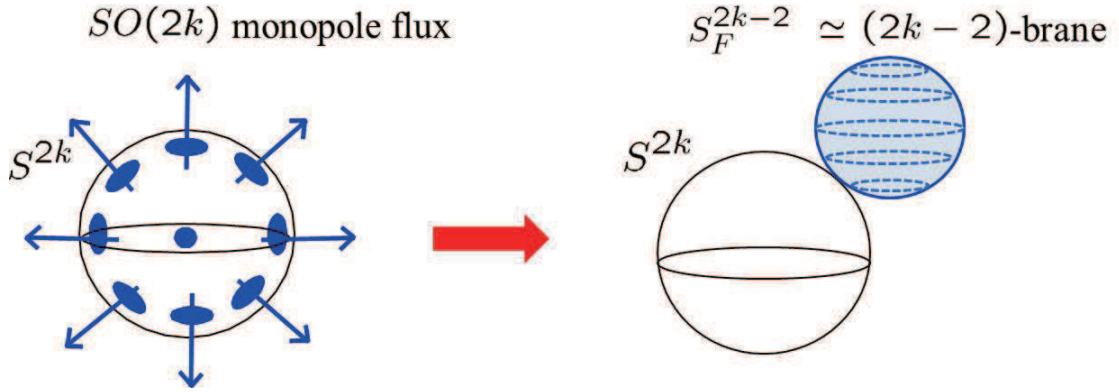


Figure 2: The internal geometry the $SO(2k)$ non-abelian flux is equivalent to the fuzzy-fibre S_F^{2k-2} , and the S_F^{2k-2} corresponds to $(2k-2)$ -brane in the enlarged $(4k-2)$ dimensional space.

4.3 The $SO(2k+1)$ Landau model

In d -dimensional space, one-particle Hamiltonian under the influence of gauge field is given by

$$H = -\frac{1}{2M} \sum_{a=1}^d D_a^2 = -\frac{1}{2M} r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + \frac{1}{2Mr^2} \sum_{a<b} \Lambda_{ab}^2, \quad (69)$$

where $D_a = \partial_a + iA_a$ and $\Lambda_{ab} = -ix_a D_b + ix_b D_a$ that satisfy

$$[\Lambda_{ab}, \Lambda_{cd}] = i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}) - i(x_a x_c F_{bd} + x_b x_d F_{ac} - x_b x_c F_{ad} - x_a x_d F_{bc}), \quad (70)$$

where F_{ab} are the components of the field strength, $F_{ab} = -i[D_a, D_b] = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$. Since the $SO(2k)$ non-abelian monopole (50) is located at the center of $d = 2k + 1$ dimensional space, its field strength is radially distributed and the system respects the $SO(2k+1)$ rotational symmetry. Hence, we have the conserved $SO(2k+1)$ angular momentum:

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab}. \quad (71)$$

It is straightforward to verify that L_{ab} act as the generators of the $SO(2k+1)$ rotation:

$$[L_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}), \quad (72)$$

where $M_{ab} = L_{ab}, \Lambda_{ab}, F_{ab}$. For a particle on $2k$ -sphere, (69) is reduced to the $SO(2k+1)$ Landau Hamiltonian:

$$H = \frac{1}{2Mr^2} \sum_{a<b} \Lambda_{ab}^2. \quad (73)$$

Due to the existence of the $SO(2k+1)$ symmetry, one may readily derive the eigenvalues of (73) by a group theoretical method. With the orthogonality $\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab}$, (73) is rewritten as

$$H = \frac{1}{2Mr^2} \left(\sum_{a<b} L_{ab}^2 - \sum_{a<b} F_{ab}^2 \right) = \frac{1}{2Mr^2} \left(\sum_{a<b} L_{ab}^2 - \sum_{\mu<\nu} \Sigma_{\mu\nu}^+{}^2 \right), \quad (74)$$

where $\sum_{a<b} F_{ab}^2 = \sum_{\mu<\nu} \Sigma_{\mu\nu}^+{}^2$ was used. We adopt the fully symmetric representation

$$(I/2) \equiv \overbrace{\left[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2} \right]}^k \quad (75)$$

for the $SO(2k)$ Casimir $\sum_{\mu<\nu} \Sigma_{\mu\nu}^+{}^2$, and the irreducible representation

$$(n, I/2) \equiv \overbrace{\left[n + \frac{I}{2}, \frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2} \right]}^k \quad (76)$$

for the $SO(2k+1)$ Casimir $\sum_{a<b} L_{ab}^2$ (n denotes the Landau level index), and the energy eigenvalues are derived as¹³

$$E_n = \frac{1}{2Mr^2} (C_{2k+1}(n, I/2) - C_{2k}(I/2)) = \frac{1}{2Mr^2} \left(n(n+2k-1) + I(n + \frac{1}{2}k) \right), \quad (78)$$

where $C_{2k+1}(n, I/2)$ and $C_{2k}(I/2)$ respectively represent the $SO(2k+1)$ and $SO(2k)$ Casimir eigenvalues for $(n, I/2)$ and $(I/2)$:

$$C_{2k+1}(n, I/2) = n^2 + n(I+2k-1) + \frac{1}{4}Ik(I+2k), \quad (79a)$$

$$C_{2k}(I/2) = \sum_{\mu<\nu} \Sigma_{\mu\nu}^{\pm 2} = \frac{1}{4}Ik(I+2k-2). \quad (79b)$$

The degeneracy in the n th Landau level is given by

$$D_n(k, I) = \frac{2n+I+2k-1}{(2k-1)!!} \frac{(n+k-1)!}{n!(k-1)!} \frac{(I+2k-3)!!}{(I-1)!!} \cdot \frac{(n+I+2k-2)!}{(n+I+k-1)!} \prod_{l=1}^{k-2} \frac{(I+2l)}{(I+l)!} \prod_{l=1}^{k-1} \frac{l!}{(2l)!}. \quad (80)$$

In particular for the lowest Landau level ($n=0$), the representation is reduced to the $SO(2k+1)$ fully symmetric spinor repr. $(I/2)$, and the degeneracy becomes to

$$D_{LLL}(k, I) = \prod_{l=1}^k \prod_{i=1}^l \frac{I+l+i-1}{l+i-1}. \quad (81)$$

In low dimensions,

$$\begin{aligned} k=1 & : D_{LLL}(1, I) = I+1, \\ k=2 & : D_{LLL}(2, I) = \frac{1}{6}(I+1)(I+2)(I+3), \\ k=3 & : D_{LLL}(3, I) = \frac{1}{360}(I+1)(I+2)(I+3)^2(I+4)(I+5), \\ k=4 & : D_{LLL}(4, I) = \frac{1}{302400}(I+1)(I+2)(I+3)^2(I+4)^2(I+5)^2(I+6)(I+7), \end{aligned} \quad (82)$$

One may notice that the lowest Landau level degeneracy (82) and the Chern number (61) are related by the following simple formula:

$$c_k(I) = D_{LLL}(k, I-1). \quad (83)$$

This relation is indeed guaranteed by the index theorem for *arbitrary* k [see Sec.4.4].

¹³ In the thermodynamic limit, $r, I \rightarrow \infty$ with I/r^2 fixed, the energy eigenvalues (78) are reduced to

$$E_n \rightarrow \frac{I}{2Mr^2} (n + \frac{1}{2}k). \quad (77)$$

The lowest Landau level energy, $E_{LLL} = \frac{I}{4Mr^2}k$, is equal to k times the lowest Landau level energy of the 2D (planar) Landau model, $\frac{B}{2M} = \frac{I}{4Mr^2}$. This is because that in the thermodynamic limit, the $2k$ D fuzzy sphere is reduced to k copies of 2D non-commutative plane.

4.4 The $SO(2k+1)$ spinor Landau model and index theorem

Here, we consider a spinor particle on S^{2k} in the $SO(2k)$ monopole background. The spinor particle carries the $SO(2k+1)$ spin degrees of freedom coupled to the external $SO(2k)$ magnetic field through Zeeman term. We analyze the $SO(2k+1)$ spinor Landau problem with use of the formulation explored by Dolan [70]. In the presence the gauge field, the Dirac operator on dD curved manifold is generally given by

$$\mathcal{D} = \gamma^\alpha D_\alpha = e^\alpha_\mu \gamma^\mu (\partial_\alpha + i\omega_\alpha + i\mathcal{A}_\alpha), \quad (84)$$

where α stand for the intrinsic coordinates of the manifold and ω_α denotes the spin connection of the manifold, while μ represent the coordinates of the dD flat Euclidean space and γ^μ stand for the $SO(d)$ gamma matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \quad (\mu, \nu = 1, 2, \dots, d) \quad (85)$$

For symmetric (\equiv torsion free) manifold, the square of the Dirac operator is given by the following Lichnerowicz formula [71]:

$$(-i\mathcal{D})^2 = -\Delta + \mathcal{F}_{\alpha\beta} \otimes \sigma^{\alpha\beta} + \frac{\mathcal{R}}{4} \quad (86)$$

where the Laplacian Δ and the field strength $\mathcal{F}_{\alpha\beta}$ are respectively given by

$$\begin{aligned} \Delta &= \frac{1}{\sqrt{g}} \nabla_\alpha (\sqrt{g} g^{\alpha\beta} \nabla_\beta) = g^{\alpha\beta} (\nabla_\alpha \nabla_\beta - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma), \\ \mathcal{F}_{\alpha\beta} &= \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha + i[\mathcal{A}_\alpha, \mathcal{A}_\beta]. \end{aligned} \quad (87)$$

and \mathcal{R} denotes the scalar curvature. The second term on the right-hand side of (86), $\sigma_{\alpha\beta} \mathcal{F}^{\alpha\beta} = e^a_\alpha e^b_\beta \sigma_{ab} \mathcal{F}^{\alpha\beta}$, represents the Zeeman term. As readily verified from the Lichnerowicz formula, in the absence of the Zeeman term, the Dirac operator does not have zero-eigenvalue on manifold with positive scalar curvature, since the eigenvalues of Laplacian are semi-positive definite. Meanwhile in the presence of the gauge field strength, the Zeeman term may cancel the contribution from the curvature term to give zero-eigenvalue for $(-i\mathcal{D})^2$. This cancellation indeed occurs in the present case, and the zero-modes of the Dirac operator are identified with the lowest Landau level basis states whose spin direction is opposite to the external magnetic field. When the gauge group is identical to the holonomy group of the coset $\mathcal{M} \simeq G/H$, (86) can be represented by the group theoretical quantities[70]:

$$(-i\mathcal{D})^2 = C(G) - C(H, R) + \frac{\mathcal{R}}{8}, \quad (88)$$

where $C(G)$ represents (quadratic) Caimir for the isometry group G and $C(H, R)$ denotes (quadratic) Casimir for the holonomy group H made by the gauge group representation R . Therefore, now we are able to derive the eigenvalues of $(-i\mathcal{D})^2$ by using a simple group theoretical method.

For $S^{2k} \simeq SO(2k+1)/SO(2k)$, we propose the $SO(2k+1)$ spinor Landau Hamiltonian as

$$H = \frac{1}{2M} (-i\mathcal{D})^2 = \frac{1}{2M} (C_{2k+1} - C_{2k}) + \frac{1}{8M} k(2k-1), \quad (89)$$

where we used the Ricci scalar of S^{2k} ¹⁴

$$\mathcal{R} = 2k(2k - 1). \quad (90)$$

For the irreducible representations

$$(n, J) \equiv [\overbrace{n + J, J, \dots, J}^k], \quad \text{for } SO(2k + 1) \quad (91a)$$

$$(\frac{I}{2}) \equiv [\overbrace{\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}}^k] \quad \text{for } SO(2k), \quad (91b)$$

the Casimir eigenvalues are respectively given by

$$C_{2k+1}(n, J) = n^2 + n(2J + 2k - 1) + kJ(J + k), \quad (92a)$$

$$C_{2k}(\frac{I}{2}) = k\frac{I}{2}(\frac{I}{2} + k - 1), \quad (92b)$$

and the eigenvalues of (89) are given by

$$E(n, J) = \frac{1}{2M}(n^2 + n(2J + 2k - 1) + k(J(J + k) - \frac{I}{2}(\frac{I}{2} + k - 1))) + \frac{1}{8M}k(2k - 1), \quad (93)$$

and the n th Landau level degeneracy is derived as

$$\begin{aligned} D_n(k, 2J) &= \frac{2n + 2J + 2k - 1}{(2k - 1)!!} \frac{(n + k - 1)!}{n!(k - 1)!} \\ &\cdot \prod_{i=1}^{k-1} (2J + 2i - 1) \cdot \prod_{i=2}^k \frac{n + 2J + 2k - i}{2k - i} \cdot \prod_{l=1}^{k-2} \prod_{i=l+2}^k \frac{2J + 2k - i - l}{2k - i - l}. \end{aligned} \quad (94)$$

For the spinor particle¹⁵, we take

$$J = \frac{I}{2} \pm \frac{1}{2}, \quad (97)$$

where for $+$ (\uparrow spin state), $I \geq 0$, while for $-$ (\downarrow spin state), $I \geq 1$. This implies that the spin polarization due to the Zeeman effect effectively changes the strength of magnetic flux by $\pm \frac{1}{2}$ according to the direction of spin. In accordance with \pm sector, (93) is block diagonalized as

$$\begin{pmatrix} E_+(n) & 0 \\ 0 & E_-(n) \end{pmatrix}, \quad (98)$$

¹⁴The $SO(2k)$ Casimir for the fundamental representation (79b) ($I = 1$) is equal to the Ricci scalar of S^{2k} ; $\sum_{\mu < \nu} \sigma_{\mu\nu}^2 = \frac{k}{4}(2k - 1) = \frac{1}{8}\mathcal{R}$.

¹⁵For the scalar particle, we substitute

$$J = \frac{I}{2} \quad (95)$$

to (89) to derive the energy eigenvalue (78):

$$H - \frac{1}{8M}k(2k - 1) = \frac{1}{2M}(C_{2k+1}(n, J) - C_{2k}(I/2))|_{J=\frac{I}{2}} = \frac{1}{2M}(n^2 + n(I + 2k - 1) + \frac{1}{2}Ik). \quad (96)$$

where $E_{\pm}(n) \equiv E(n, J)_{J=\frac{I}{2} \pm \frac{1}{2}}$:

$$\begin{aligned} E_+(n) &= \frac{1}{2M}(n^2 + n(I + 2k) + k(I + k)), \\ E_-(n) &= \frac{1}{2M}(n^2 + n(I + 2k - 2)), \end{aligned} \quad (99)$$

whose degeneracies are respectively given by $D_n(k, I + 1)$ and $D_n(k, I - 1)$ through the formula (94)¹⁶. In low dimensions, (99) reads as

$$\begin{aligned} S^2 : \quad & \begin{pmatrix} E_+(n) & 0 \\ 0 & E_-(n) \end{pmatrix} \Big|_{k=1} = \frac{1}{2M} \begin{pmatrix} (n+1)(n+I+1) & 0 \\ 0 & n(n+I) \end{pmatrix}, \\ S^4 : \quad & \begin{pmatrix} E_+(n) & 0 \\ 0 & E_-(n) \end{pmatrix} \Big|_{k=2} = \frac{1}{2M} \begin{pmatrix} n^2 + n(I+4) + 2(I+2) & 0 \\ 0 & n^2 + n(I+2) \end{pmatrix}, \\ S^6 : \quad & \begin{pmatrix} E_+(n) & 0 \\ 0 & E_-(n) \end{pmatrix} \Big|_{k=3} = \frac{1}{2M} \begin{pmatrix} n^2 + n(I+6) + 3(I+3) & 0 \\ 0 & n^2 + n(I+4) \end{pmatrix}, \\ S^8 : \quad & \begin{pmatrix} E_+(n) & 0 \\ 0 & E_-(n) \end{pmatrix} \Big|_{k=4} = \frac{1}{2M} \begin{pmatrix} n^2 + n(I+8) + 4(I+4) & 0 \\ 0 & n^2 + n(I+6) \end{pmatrix}. \end{aligned} \quad (101)$$

The Landau level energy spectrum is bounded by zero for the lowest Landau level basis states ($n = 0$) with \downarrow spin:

$$E_-(n = 0) = 0, \quad (102)$$

and the number of the zero-energy states is given by

$$D_{LLL}(k, I - 1). \quad (103)$$

Since the Hamiltonian is the square of the Dirac operator, the zero-energy eigenstates correspond to the zero-modes of the Dirac operator:

$$\text{Ind}(i\mathcal{D}) = D_{LLL}(k, I - 1), \quad (104)$$

and the index theorem tells that the number of zero-modes is equal to the topological charge of the non-trivial gauge configuration:

$$\text{Ind}(i\mathcal{D}) = c_k, \quad (105)$$

where c_k denotes the k th Chern number of the $SO(2k)$ monopole (59). Thus, we verified (83) for arbitrary k .

¹⁶ It can be confirmed that $E_+(n)|_{I=0}$ (99) and $D_n(k, 2J = I + 1)|_{I=0} = D_n(k, 1)$ (94) respectively reproduce the eigenvalues and the degeneracy of the free Dirac operator without gauge field [72, 73, 74, 75]:

$$\begin{aligned} \sqrt{2ME_+(n)}|_{I=0} &= n + k, \\ D_n(k, 1) &= 2^k \binom{n + 2k - 1}{n}. \end{aligned} \quad (100)$$

4.5 Laughlin-like wavefunction

For higher dimensional quantum Hall effect, the particles carry the $SO(2k)$ color degrees of freedom with the geometry S_F^{2k-2} , and the total space will be given by

$$(\mathbf{x}, \mathbf{y}) \in S^{2k} \times S^{2k-2}, \quad (106)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{2k+1})$ with $\sum_{a=1}^{2k+1} x_a x_a = r^2$ denotes the basemanifold S^{2k} while $\mathbf{y} = (y_1, y_2, \dots, y_{2k-2})$ with $\sum_{i=1}^{2k-1} y_i y_i = r^2$ represents the coordinates on $(2k-2)$ -dimensional internal space S^{2k-2} (which is regarded as the classical counterpart of fuzzy bundle coordinates X_i (30)). The coordinates of the total space $S^{2k} \otimes S^{2k-2}$ is represented by

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{2r(r+x_{2k+1})}} \begin{pmatrix} (r+x_{2k+1})\psi \\ (x_{2k} + i\gamma_i x_i)\psi \end{pmatrix}. \quad (107)$$

where ψ denotes 2^{k-1} component spinor giving the internal coordinates by the relation:

$$\psi^\dagger \gamma_i \psi = y_i. \quad (108)$$

The lowest Landau level basis states can be constructed by taking a fully symmetric products of the components of $\Psi(\mathbf{x})$:

$$\Psi_{m_1, m_2, \dots, m_{2k}}(\mathbf{x}) = \frac{1}{\sqrt{m_1! m_2! \dots m_{2k}!}} \Psi_1^{m_1}(\mathbf{x}) \Psi_2^{m_2}(\mathbf{x}) \dots \Psi_{2k}^{m_{2k}}(\mathbf{x}), \quad (109)$$

with $m_1 + m_2 + \dots + m_{2k} = I$. For $m = 1$ the particles occupy all the lowest Landau level states on S^{2k} , and so the total particle number N is

$$N \equiv d(k, I) \equiv \frac{D(k, I)}{D(k-1, I)} = \frac{(k-1)!}{(2k-1)!} \frac{(I+2k-1)!}{(I+k-1)!} \sim I^k, \quad (110)$$

where $D(k, I)$ denotes the number of states of the total space S_F^{2k} , and $D(k-1, I)$ stands for the number of states of the fuzzy-fibre S_F^{2k-2} . For $I/2 \rightarrow mI/2$, the state number on S^{2k} changes as

$$d(k, mI) = \frac{D(k, mI)}{D(k-1, mI)} = \frac{(k-1)!}{(2k-1)!} \frac{(mI+2k-1)!}{(mI+k-1)!} \sim (mI)^k. \quad (111)$$

With use of the Slater determinant, the Laughlin-like groundstate wavefunction is constructed as

$$\Psi_{\text{Lin}}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \prod_{i < j=1}^{N=d(k, I)} (\epsilon_{A_1 A_2 \dots A_N} \Psi_{A_1}(\mathbf{x}_1) \Psi_{A_2}(\mathbf{x}_2) \dots \Psi_{A_N}(\mathbf{x}_N))^m, \quad (112)$$

where $A = (m_1, m_2, \dots, m_{2k})$ and m is taken as odd integer to keep the Fermi statistics of the particles. When the power of Ψ_A changes from 1 to m , the monopole charge changes from I to mI , and then Ψ_{Lin} is considered to correspond to the $2k$ D quantum Hall liquid at the filling factor:

$$\nu_{2k} = \frac{N}{d(k, mI)} \simeq \frac{1}{m^k}. \quad (113)$$

Notice that since m is an odd inter, ν_{2k} is also the inverse of an odd integer. From the perspective of the original basemanifold S^{2k} , Ψ_{Lin} denotes the incompressible liquid made of the particles. However, from the emergent $(4k-1)$ D space-time point of view, the particle corresponds to $(2k-2)$ -brane, and Ψ_{Lin} is alternatively interpreted as a many-body state of membranes.

5 Tensor Monopole Fields from Non-Abelian Monopole Fields

We discussed the non-abelian monopoles whose gauge group is compatible with the holonomy of sphere. In this section, we introduce the other type of monopole, the tensor monopole [40, 41] whose gauge group is $U(1)$ whole gauge field is not a vector field but an antisymmetric tensor field¹⁷.

5.1 Tensor monopole fields

To begin with, we review several basic properties of the n -form tensor gauge field [41]:

$$C_n = \frac{1}{n!} C_{a_1 a_2 \dots a_n} dx^{a_1} dx^{a_2} \dots dx^{a_n} \quad (114)$$

where $C_{a_1 a_2 \dots a_n}$ represent a totally antisymmetric tensor gauge field. Notice that $C_{a_1 a_2 \dots a_n}$ is *not* a matrix-valued gauge field but a tensor extension of the $U(1)$ gauge field. Like the ordinary $U(1)$ gauge theory, the field strength is defined as

$$G_{n+1} = dC_n = \frac{1}{(n+1)!} G_{a_1 a_2 \dots a_{n+1}} dx^{a_1} dx^{a_2} \dots dx^{a_{n+1}}, \quad (115)$$

where

$$G_{a_1 a_2 \dots a_{n+1}} = \frac{1}{n!} \partial_{[a_1} C_{a_2 \dots a_{n+1}]} \quad (116)$$

For instance,

$$\begin{aligned} n = 2 : \quad G_{abc} &= \partial_a C_{bc} + \partial_b C_{ca} + \partial_c C_{ab}, \\ n = 3 : \quad G_{abcd} &= \partial_a C_{bcd} - \partial_b C_{cda} + \partial_c C_{dab} - \partial_d C_{abd}. \end{aligned} \quad (117)$$

The $U(1)$ gauge symmetry is incorporated in the following way. The $U(1)$ gauge transformation is given by

$$C_n \rightarrow C_n + d\Lambda_{n-1}, \quad (118)$$

with

$$\Lambda_{n-1} = \frac{1}{(n-1)!} \Lambda_{a_1 a_2 \dots a_{n-1}} dx_{a_1} dx_{a_2} \dots dx_{a_{n-1}}. \quad (119)$$

It is obvious that the field strength G is invariant under (118). In terms of the tensor components, the gauge transformation is represented as

$$C_{a_1 a_2 \dots a_n} \rightarrow C_{a_1 a_2 \dots a_n} + \frac{1}{(n-1)!} \partial_{[a_1} \Lambda_{a_2 \dots a_n]}. \quad (120)$$

For instance,

$$\begin{aligned} n = 2 : \quad C_{ab} &\rightarrow C_{ab} + \partial_a \Lambda_b - \partial_b \Lambda_a, \\ n = 3 : \quad C_{abc} &\rightarrow C_{abc} + \partial_a \Lambda_{bc} + \partial_b \Lambda_{ca} + \partial_c \Lambda_{ab}. \end{aligned} \quad (121)$$

¹⁷The antisymmetric tensor gauge field is realized as a solution of the Kalb-Ramond equation and also referred to as the Kalb-Ramond field [42].

/	Non-abelian monopole	Tensor monopole
Sphere	S^{2k}	S^{2k}
Gauge group	$SO(2k)$	$U(1)$
Rank of gauge field	1	$2k - 1$
Rank of field strength	2	$2k$

Table 1: Relations between the non-abelian monopole and the tensor monopole.

It is a simple exercise to see that (117) is invariant under (121). The field strength of the $U(1)$ tensor monopole located at the origin of $(n + 2)$ D Euclidean space is given by

$$G_{a_1 a_2 \dots a_{n+1}} = g \frac{1}{r^{n+2}} \epsilon_{a_1 a_2 \dots a_{n+2}} x_{a_{n+2}}, \quad (122)$$

where g denotes the charge of $U(1)$ tensor monopole. The integral of the gauge field strength over S^n yields

$$\int_{S^{n+1}} G_{n+1} = g \mathcal{A}(S^{n+1}), \quad (123)$$

where $\mathcal{A}(S^{n+1})$ represents the area of S^{n+1} .

5.2 Relation between field strengths of monopoles

The non-abelian and tensor monopoles are two different extensions of the Dirac monopole in terms of internal and external indices respectively. As discussed in Sec.3, there is no reasonable distinction between the external and internal spaces in the lowest Landau level, and so it is expected that non-abelian and tensor monopoles should be “equivalent” in some sense. Interestingly, for the $SU(2)$ monopole and 3-rank tensor monopole, their connection has already been pointed out, at least for fundamental representation (quaternions) [76] and for the integral form [77]. As a natural generalization of these results, we demonstrate connection between tensor and non-abelian monopoles for fully symmetric representation in arbitrary even dimensions. In the following, we take n as an odd integer, $n = 2k - 1$ and the monopole at the center of S^{2k} [see Table 1] and consider the tensor monopole gauge field of the following form

$$G_{a_1 a_2 \dots a_{2k}} = g_k \frac{1}{r^{2k+1}} \epsilon_{a_1 a_2 \dots a_{2k+1}} x_{a_{2k+1}}. \quad (124)$$

We fix the ratio between two monopole charges, c_k and g_k , by imposing the condition:

$$\int_{S^{2k}} G_{2k} = \text{tr} \int_{S^{2k}} F^k. \quad (125)$$

From

$$\int_{S^{2k}} G_{2k} = g_k \mathcal{A}(S^{2k}) \quad (126)$$

with

$$\mathcal{A}(S^{2k}) = \frac{2^{k+1} \pi^k}{(2k - 1)!!}, \quad (127)$$

and

$$\int_{S^{2k}} \text{tr} F^k = (2k)!! \pi^k c_k, \quad (128)$$

the relation between two monopole charges is determined as

$$g_k = \frac{(2k)!}{2^{k+1}} c_k. \quad (129)$$

Eq.(125) is rather “trivial”, since we are always able to impose (125) by fixing the ratio between the two monopole charges. What we really need to verify is the *local* non-abelian and tensor monopole relation:

$$G_{2k} = \text{tr} F^k. \quad (130)$$

To prove (130) we use a brute force method: We substitute the explicit form of F (54) to the right-hand side of (130) to see whether we can derive G (124) on the left-hand side under the identification (129). For the component relation between $G_{a_1 a_2 \dots a_{2k}}$ ($a_1, a_2, \dots, a_{2k} = 1, 2, \dots, 2k+1$) and F_{ab} , the local relation (130) can be rewritten as¹⁸.

$$G_{a_1 a_2 \dots a_{2k}} = \frac{1}{2^k} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_{2k}} a_{2k+1}} \text{tr}(F_{b_{a_1} b_{a_2}} \dots F_{b_{a_{2k-1}} b_{a_{2k}}}). \quad (133)$$

For instance,

$$G_{12 \dots 2k} = \frac{1}{2^k} \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} \text{tr}(F_{\mu_1 \mu_2} \dots F_{\mu_{2k-1} \mu_{2k}}), \quad (134)$$

where $\mu_1, \mu_2, \dots, \mu_{2k} = 1, 2, \dots, 2k$. We substitute (54) to the right-hand side of (134) and perform a straightforward calculation with use of the formulae for the $SO(2k)$ matrices (243), and then we find the right-hand side of (134) gives

$$G_{12 \dots 2k} = \frac{(2k)!}{2^{k+1} r^{2k+1}} x_{2k+1}. \quad (135)$$

In the covariant notation, (135) is expressed as

$$G_{a_1 a_2 \dots a_{2k}} = \frac{(2k)!}{2^{k+1} r^{2k+1}} \epsilon_{a_1 a_2 \dots a_{2k+1}} x_{a_{2k+1}} \quad (136)$$

or

$$G_{2k} = \frac{1}{2^{k+1} r^{2k+1}} \epsilon_{a_1 a_2 \dots a_{2k+1}} x_{a_{2k+1}} dx_{a_1} dx_{a_2} \dots dx_{a_{2k}}. \quad (137)$$

¹⁸ Here, we used

$$G_{2k} = \frac{1}{(2k)!} G_{a_1 a_2 \dots a_{2k}} dx_{a_1} dx_{a_2} \dots dx_{a_{2k}}. \quad (131)$$

and

$$\text{tr} F^k = \frac{1}{2^k} \text{tr}(F_{a_1 a_2} \dots F_{a_{2k-1} a_{2k}}) dx_{a_1} dx_{a_2} \dots dx_{a_{2k}}. \quad (132)$$

For instance,

$$\begin{aligned}
U(1) : \quad G_{ij} &= \frac{1}{2r^3} \epsilon_{ijk} x_k, & (i, j, k = 1, 2, 3) \\
SU(2) : \quad G_{abcd} &= \frac{3}{r^5} \epsilon_{abcde} x_e, & (a, b, c, d, e = 1, 2, 3, 4, 5) \\
SO(6) : \quad G_{a_1 a_2 \dots a_6} &= \frac{45}{r^7} \epsilon_{a_1 a_2 \dots a_6 a_7} x_{a_7}, & (a_1, a_2, \dots, a_7 = 1, 2, \dots, 7) \\
SO(8) : \quad G_{a_1 a_2 a_3 \dots a_8} &= \frac{1260}{r^9} \epsilon_{a_1 a_2 \dots a_8 a_9} x_{a_9}. & (a_1, a_2, \dots, a_9 = 1, 2, \dots, 9)
\end{aligned} \tag{138}$$

Thus we derived the tensor monopole gauge field G from $\text{tr} F^k$. Furthermore, for a general symmetric representation of the $SO(2k)^{19}$

$$(I/2) \equiv \overbrace{\left[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2} \right]}^k,$$

we have a generic expression for the $U(1)$ antisymmetric tensor field strength as

$$\begin{aligned}
G_{a_1 a_2 \dots a_{2k}} &= \frac{(2k)! I}{2^{k+2}} C(k, I) D(k-1, I) \frac{1}{r^{2k+1}} \epsilon_{a_1 a_2 \dots a_{2k+1}} x_{a_{2k+1}} \\
&= \frac{I}{2} C(k, I) D(k-1, I) G_{a_1 a_2 \dots a_{2k}}^{(I=1)},
\end{aligned} \tag{139}$$

where $C(k, I)$ and $G_{a_1 a_2 \dots a_{2k+1}}^{(I=1)}$ are respectively given by (34) and (136). Here, we used the formulae for the symmetric representation (244). It is rather simple to confirm the symmetric representation (139) for $I = 1$ reproduces (136) by the formula

$$D_{LLL}(k, I = 1) = \frac{(2k)!!}{k!} = 2^k. \tag{140}$$

With (140) and the formula about the lowest Landau level degeneracy

$$C(k, I) D_{LLL}(k-1, I) = \frac{(2k)!}{2^k I} D_{LLL}(k, I-1), \tag{141}$$

we finally find that G takes an amazingly simple form²⁰:

$$G_{a_1 a_2 \dots a_{2k}} = c_k(I) \cdot G_{a_1 a_2 \dots a_{2k}}^{(I=1)}, \tag{144}$$

¹⁹ $I = 1$ corresponds to the spinor representation.

²⁰In differential form, (144) is represented as

$$G_{2k} = \frac{1}{2^{k+1} r^{2k+1}} c_k(I) \epsilon_{a_1 a_2 \dots a_{2k+1}} x_{a_{2k+1}} dx_{a_1} dx_{a_2} \dots dx_{a_{2k}} = c_k(I) G_{2k}^{(I=1)}, \tag{142}$$

and hence the normalized $U(1)$ tensor monopole charge $q_k(I) \equiv \frac{1}{\int_{S^{2k}} G_{2k}^{(I=1)}} \int_{S^{2k}} G_{2k}$, is identical to the Chern number:

$$q_k(I) = c_k(I). \tag{143}$$

where $G^{(I=1)}$ is given by (136) and the relation (83) was used. From (144), we can read off the tensor monopole charge g_k as $g_k = \frac{(2k)!}{2^{k+1}} c_k(I)$, which is consistent with the result (129). In low dimensions, we have

$$\begin{aligned} G_{ij} &= \frac{1}{2r^3} I \epsilon_{ijk} x_k, \\ G_{abcd} &= \frac{1}{2r^5} I(I+1)(I+2) \epsilon_{abcde} x_e, \\ G_{a_1 a_2 \dots a_6} &= \frac{1}{8r^7} I(I+1)(I+2)^2(I+3)(I+4) \epsilon_{a_1 a_2 \dots a_7} x_{a_7}, \\ G_{a_1 a_2 a_3 \dots a_8} &= \frac{1}{240r^9} I(I+1)(I+2)^2(I+3)^2(I+4)^2(I+5)(I+6) \epsilon_{a_1 a_2 \dots a_9} x_{a_9}. \end{aligned} \quad (145)$$

Thus, we verified the local non-abelian and tensor monopole relation (130) for generic fully symmetric representation in arbitrary even dimensions.

5.3 Relation between gauge fields of monopoles

For non-abelian gauge field, we have [78]

$$\text{tr}(F^k) = dL_{CS}^{(2k-1)}[A], \quad (146)$$

where $L_{CS}^{(2k-1)}$ represents the Chern-Simons term

$$L_{CS}^{(2k-1)}[A] = k \int_0^1 dt \text{tr}(A(tdA + it^2 A^2)^{k-1}). \quad (147)$$

Meanwhile for the tensor monopole gauge field, we have seen

$$G_{2k} = dC_{2k-1}. \quad (148)$$

From the non-abelian and tensor monopole relation (130), it is obvious that the tensor monopole gauge field is identical to the Chern-Simons term of the non-abelian gauge field:

$$C_{2k-1} = \text{tr}(L_{CS}^{(2k-1)}[A]). \quad (149)$$

For instance,

$$\begin{aligned} C_1 &= \text{tr} A, \\ C_3 &= \text{tr}(AdA + \frac{2}{3}iA^3) = \text{tr}(AF - \frac{1}{3}iA^3), \\ C_5 &= \text{tr}(A(dA)^2 + \frac{3}{2}iA^3dA - \frac{3}{5}A^5) = \text{tr}(AF^2 - \frac{1}{2}iA^3F - \frac{1}{10}A^5), \\ C_7 &= \text{tr}(A(dA)^3 + \frac{8}{5}iA^3(dA)^2 + \frac{4}{5}iA(AdA)^2 - 2A^5dA - \frac{4}{7}iA^7) \\ &= \text{tr}(AF^3 - \frac{2}{5}iA^3F^2 - \frac{1}{5}iAF A^2F - \frac{1}{5}A^5F + \frac{1}{35}iA^7). \end{aligned} \quad (150)$$

Notice that $\text{tr}(A^3 F^2) \neq \text{tr}(AF A^2 F)$, since A and F take their values in matrix and are not commutative. For components of (150), we have

$$\begin{aligned}
C_i &= \text{tr} A_i, \\
C_{abc} &= \text{tr}(A_{[a} \partial_b A_{c]} + \frac{2}{3} i A_{[a} A_b A_{c]}) = \frac{1}{2} \text{tr}(A_{[a} F_{bc]} - \frac{2}{3} i A_{[a} A_b A_{c]}), \\
C_{abcde} &= \frac{1}{4} \text{tr}(A_{[a} F_{bc} F_{de]} - i A_{[a} A_b A_c F_{de]} - \frac{2}{5} A_{[a} A_b A_c A_d A_{e]}), \\
C_{a_1 a_2 \dots a_7} &= \frac{1}{8} \text{tr}(A_{[a_1} F_{a_2 a_3} F_{a_4 a_5} F_{a_6 a_7]} - \frac{4}{5} i A_{[a_1} A_{a_2} A_{a_3} F_{a_4 a_5} F_{a_6 a_7]} - \frac{2}{5} i A_{[a_1} F_{a_2 a_3} A_{a_4} A_{a_5} F_{a_6 a_7]} \\
&\quad - \frac{4}{5} A_{[a_1} A_{a_2} A_{a_3} A_{a_4} A_{a_5} F_{a_6 a_7]} + \frac{8}{35} i A_{[a_1} A_{a_2} A_{a_3} A_{a_4} A_{a_5} A_{a_6} A_{a_7]}). \tag{151}
\end{aligned}$$

The $SO(2k)$ gauge transformation for A acts as the $U(1)$ gauge transformation for C_{2k-1} . For instance $k = 2$, the non-abelian ($SU(2)$) gauge transformation (57) acts to C_3 as

$$C_3 \rightarrow C_3 - id(\text{tr} A d g g^\dagger) + \frac{1}{3} \text{tr}(g^\dagger d g)^3. \tag{152}$$

The second term on the right-hand side is the total derivative. The third term satisfies²¹

$$d(\text{tr}(g^\dagger d g)^3) = -\text{tr}(g^\dagger d g)^4 = 0, \tag{153}$$

and is locally expressed as a total derivative (Poincaré Lemma). Consequently, (152) can be rewritten in the following form

$$C_3 \rightarrow C_3 + d\Lambda_2. \tag{154}$$

In general, the $SO(2k)$ gauge transformation acts as $U(1)$ gauge transformation to tensor gauge field (see Appendix C for more details):

$$C_{2k-1} \rightarrow C_{2k-1} + d\Lambda_{2k-2}. \tag{155}$$

For practical applications, it is important to derive the explicit form of the tensor monopole gauge field. From the general formula (151), we derive the tensor monopole gauge field from the non-abelian monopole in low dimensions. We substitute the non-abelian monopole field (50) to the right-hand side of the formula (151). After a long straightforward calculation with use of trace formulae for gamma matrices, we obtain the following expressions for spinor representation:

$$\begin{aligned}
C_i &= -\frac{1}{2r(r+x_3)} \epsilon_{ij3} x_j, \\
C_{abc} &= -\frac{1}{r^3} \left(\frac{1}{r+x_5} + \frac{r}{(r+x_5)^2} \right) \epsilon_{abcd5} x_d, \\
C_{abcde} &= -\frac{9}{r^5} \left(\frac{1}{r+x_7} + \frac{r}{(r+x_7)^2} + \frac{2}{3} \frac{r^2}{(r+x_7)^3} \right) \epsilon_{abcdef7} x_f, \\
C_{a_1 a_2 \dots a_7} &= -\frac{180}{r^7} \left(\frac{1}{r+x_9} + \frac{r}{(r+x_9)^2} + \frac{4}{5} \frac{r^2}{(r+x_9)^3} + \frac{2}{5} \frac{r^3}{(r+x_9)^4} \right) \epsilon_{a_1 a_2 \dots a_8 9} x_{a_8}. \tag{156}
\end{aligned}$$

²¹ $\text{tr}(\alpha^{2n}) = 0$ for any one-form $\alpha = dx_a \alpha_a$.

Notice that $(2k - 1)$ rank tensor monopole gauge field exhibits k th power string-like singularity. Similarly for fully symmetric representation, we obtain

$$\begin{aligned}
C_i &= -\frac{I}{2r(r+x_3)}\epsilon_{ij3}x_j, \\
C_{abc} &= -\frac{1}{6r^3}I(I+1)(I+2)\left(\frac{1}{r+x_5} + \frac{r}{(r+x_5)^2}\right)\epsilon_{abcd5}x_d, \\
C_{abcde} &= -\frac{1}{40r^5}I(I+1)(I+2)^2(I+3)(I+4)\left(\frac{1}{r+x_7} + \frac{r}{(r+x_7)^2} + \frac{2}{3}\frac{r^2}{(r+x_7)^3}\right)\epsilon_{abcdef7}x_f \\
C_{a_1a_2\cdots a_7} &= -\frac{1}{1680r^7}I(I+1)(I+2)^2(I+3)^2(I+4)^2(I+5)(I+6) \\
&\quad \times \left(\frac{1}{r+x_9} + \frac{r}{(r+x_9)^2} + \frac{4}{5}\frac{r^2}{(r+x_9)^3} + \frac{2}{5}\frac{r^3}{(r+x_9)^4}\right)\epsilon_{a_1a_2\cdots a_89}x_{a_8}. \quad (157)
\end{aligned}$$

For $I = 1$, (157) is reduced to (156). One may also confirm that (157) indeed gives the field strength(145) through the formula:

$$G_{a_1a_2\cdots a_{2k}} = \frac{1}{(2k-1)!}\partial_{[a_1}C_{a_2\cdots a_{2k-1]}}. \quad (158)$$

5.4 Quantum Nambu geometry via tensor monopole

In the lowest Landau level, the covariant angular momentum is quenched, and then we have the identification:

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab} \sim r^2 F_{ab}. \quad (159)$$

In 3D, two rank antisymmetric tensor is equivalent to vector, and the angular momentum is directly related to the coordinates of fuzzy two-sphere (24). However in higher dimensions, two rank antisymmetric tensor is no longer equivalent to vector and the angular momentum is not apparently related to the coordinates of fuzzy sphere. As mentioned in Sec.3.2, the quantum Nambu bracket suggests the existence of tensor monopole, and we have shown the non-abelian and tensor monopole relation (130) or

$$\frac{1}{r^{2k+1}}x_a = \frac{2}{(2k)!c_k}\epsilon_{aa_1a_2\cdots a_{2k}}\text{tr}(F_{a_1a_2}\cdots F_{a_{2k-1}a_{2k}}). \quad (160)$$

The identification (159) suggests that (160) becomes to

$$X_a = \frac{I}{(2k)!c_k} \propto \epsilon_{aa_1a_2\cdots a_{2k}}(L_{a_1a_2}L_{a_3a_4}\cdots L_{a_{2k-1}a_{2k}}) \quad (161)$$

in the lowest Landau level, and the coordinates of higher dimensional sphere are now regarded as the operators. (161) is a natural generalization of (25).

Let us consider the algebra for X_a . For this purpose, it is useful to adopt the analogy between the algebras of X_a and the covariant derivatives $-iD_a$ [30]. For S_F^2 case, the algebra of X_i is given by

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k, \quad (162)$$

while the covariant derivative gives

$$[-iD_i, -iD_j] = -iF_{ij} = -i\frac{1}{\alpha r^2}\epsilon_{ijk}x_k. \quad (163)$$

One may notice the analogy:

$$[X_i, X_j] \leftrightarrow -(\alpha r)^2[-iD_i, -iD_j]. \quad (164)$$

This analogy can hold in higher dimensions [see Sec.3.2], and for evaluation of the Nambu bracket for X_a we utilize the following identification:

$$[X_{a_1}, X_{a_2}, \dots, X_{a_{2k}}] \leftrightarrow \frac{1}{D_{LLL}(k-1, I)}(-(\alpha r)^2)^k[-iD_{a_1}, -iD_{a_2}, \dots, -iD_{a_{2k}}]. \quad (165)$$

The right-hand side gives

$$\begin{aligned} & [-iD_{a_1}, -iD_{a_2}, \dots, -iD_{a_{2k}}] \\ &= \frac{1}{2^k} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_{2k}} a_{2k+1}} [-iD_{b_{a_1}}, -iD_{b_{a_2}}] [-iD_{b_{a_3}}, -iD_{b_{a_4}}] \dots [-iD_{b_{a_{2k-1}}}, -iD_{b_{a_{2k}}}] \\ &= (-i\frac{1}{2})^k \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_{2k}} a_{2k+1}} F_{b_{a_1} b_{a_2}} F_{b_{a_3} b_{a_4}} \dots F_{b_{a_{2k-1}} b_{a_{2k}}}, \end{aligned} \quad (166)$$

and the trace is evaluated as

$$\text{tr}[-iD_{a_1}, -iD_{a_2}, \dots, -iD_{a_{2k}}] = (-i)^k \frac{(2k)!}{2^{k+1}} D_{LLL}(k, I-1) \cdot \epsilon_{a_1 a_2 \dots a_{2k+1}} \frac{1}{r^{2k+1}} x_{a_{2k+1}}. \quad (167)$$

Due to the relation (141), we finally obtain

$$[X_{a_1}, X_{a_2}, \dots, X_{a_{2k}}] = i^k C(k, I) \alpha^{2k-1} \epsilon_{a_1 a_2 \dots a_{2k+1}} X_{a_{2k+1}}, \quad (168)$$

which is exactly equal to the quantum Nambu algebra for fuzzy sphere (33).

6 Flux Attachment and Tensor Chern-Simons Field Theory for Membranes

Here we discuss physics of A-class topological insulator based on Chern-Simons tensor field theory. We will see exotic concepts in 2D quantum Hall effect are naturally generalized in higher dimensions:

- Flux attachment and composite particles [45, 46, 44]
- Effective topological field theory [43, 44]
- Fractional statistics of quasi-particle excitations [79]
- Haldane-Halperin hierarchy [28, 80]
- \vdots

6.1 Basic observations

Before going to the details, we summarize basic observations about the relevant physical concepts and associated mathematics in higher dimensions.

- $(2k - 1)$ rank tensor gauge field and $(2k - 2)$ -brane

The $(2k - 1)$ rank gauge field is naturally coupled to the $(2k - 1)$ rank current of $(2k - 2)$ -brane. The membrane degrees of freedom is *automatically* incorporated in the geometry of S_F^{2k} as the fuzzy fibre S_F^{2k-2} over S^{2k} ;

$$S_F^{2k} \simeq S^{2k} \otimes S_F^{2k-2}. \quad (169)$$

Although S_F^{2k-2} represents the internal non-abelian gauge space of the particle, the internal space is as large as the external space S^{2k} , and it can be regarded as $(2k - 2)$ -brane in the enlarged space [see Fig.2] that consists of the external S^{2k} and the “internal” S^{2k-2} which membrane occupies. Since membrane is associated with the non-abelian flux of non-abelian monopole, membrane can be considered as a charged excitation induced by a penetration of the non-abelian flux in higher dimensions.

- Emergence of $(4k - 1)$ D space-time and J -homomorphism

Though we started from the $(2k + 1)$ D space-time where particles with color degrees of freedom and the $SO(2k)$ non-abelian monopole live, we arrive at $(4k - 1)$ D space-time where $(2k - 2)$ -brane and $(2k - 1)$ rank tensor monopole live. Mathematically, the Hopf-Whitehead J -homomorphism [81, 76, 86, 88]²² accounts for the intimate connection between the $(2k + 1)$ D space(-time) and the $(4k - 1)$ D space(-time):

$$\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z} \rightarrow \pi_{4k-1}(S^{2k}) \simeq \mathbb{Z}. \quad (173)$$

The left homotopy is related to the $SO(2k)$ monopole at the origin of $(2k + 1)$ D space and describes the non-trivial winding from the equator of S^{2k} to the $SO(2k)$ monopole gauge group, while the right homotopy describes a non-trivial winding from $(4k - 1)$ space(-time) to the base-manifold S^{2k} on which $(2k - 2)$ -brane lives. In particular for $k = 1$, (173) gives

$$\pi_1(SO(2)) \simeq U(1) \simeq \mathbb{Z} \rightarrow \pi_3(S^2) \simeq \mathbb{Z}. \quad (174)$$

²² In general, J -homomorphism represents the homomorphism between the homotopy group of the orthogonal group and that of sphere:

$$\pi_l(SO(M)) \rightarrow \pi_{l+M}(S^M). \quad (170)$$

Eq.(173) is a special case of (170) for $l = 2k - 1$ and $M = 2k$. When $l = 1$, the homomorphism (170) becomes the isomorphism:

$$\pi_1(SO(M)) = \pi_{M+1}(S^M), \quad (171)$$

which gives the 1st Hopf map, $\pi_3(S^2) = \pi_1(SO(2)) = \mathbb{Z}$, for $M = 2$. The other two Hopf maps are also obtained as the J -homomorphism (173) for $k = 2, 4$:

$$\begin{aligned} \pi_3(SO(4)) = \mathbb{Z} + \mathbb{Z} &\rightarrow \pi_7(S^4) = \mathbb{Z} + \mathbb{Z}_{12}, \\ \pi_7(SO(8)) = \mathbb{Z} + \mathbb{Z} &\rightarrow \pi_{15}(S^8) = \mathbb{Z} + \mathbb{Z}_{120}. \end{aligned} \quad (172)$$

Dim.	0	1	2	\dots	$2k-2$	$2k-1$	$2k-2$	\dots	$4k-4$	$4k-3$	$4k-2$
M_{2k-2}	\circ	\circ	\circ	\dots	\circ						
M_{2k-2}	\circ					\circ	\circ	\dots	\circ		

Table 2: Two $(2k-2)$ -branes in $(4k-1)$ D space-time. From the co-dimension 2 space, the membranes are regarded as “point-particles”.

The left homotopy guarantees the non-trivial topology of Dirac monopole bundle, while the right homotopy is the 1st Hopf map which represents the underlying mathematics of fractional statistics for 0-brane in 3D space(-time) [82]. The world line of the 0-brane on S^2 corresponds to the S^1 fibre on S^2 , and the non-trivial linking of world lines of two 0-branes indicates the topological number denoted by the 1st Hopf map [84]. Similarly, the non-trivial homotopy $\pi_{4k-1}(S^{2k}) \simeq \mathbb{Z}$ is related to the fractional statistics in $(4k-1)$ D space(-time) [85, 86, 87]. The dimension of the object obeying the fractional statistics can readily be obtained by the following dimensional counting. Since the dimension of the total space(-time) is $(4k-1)$ and S^{2k} is the basemanifold, the remaining $(4k-1)-2k=2k-1$ dimension should be the dimension of the world volume of the object that obeys the fractional statistics. Indeed the dimension of $(2k-2)$ -brane world volume is $(2k-1)$ dimension, and so $(2k-2)$ -branes are expected to obey the fractional statistics.

Another way to see $(2k-2)$ -brane can obey fractional statistics is to notice the co-dimension. The necessary condition for the existence of fractional statistics is the co-dimension 2 where the braiding operation has non-trivial meaning. Indeed, the co-dimension of two (non-overlapping) $(2k-2)$ -branes in $(4k-2)$ space is 2 [Table 2]. From the co-dimension, two membranes are regarded as two point-like objects, and the idea of fractional statistics (for point-like object) in 3D can similarly be applied to higher dimensions.

- Physical realization of fractional statistics

The statistical transformation is physically achieved by acquiring Aharonov-Bohm phase [89, 90], where the particles acquire a statistical phase during a trip around the magnetic flux. In the fractional quantum Hall effect, the statistical phase accounts for the fractional statistics of fractionally charged quasi-particle excitation [79] and also for the statistical transformation from electron to composite boson at the odd-denominator fillings [45, 46, 44]. The statistical transformation to composite boson is elegantly described by the Chern-Simons field theory formulation [43, 44]. In higher dimensions, there are $(2k-2)$ -branes coupled to the $(2k-1)$ rank tensor $U(1)$ gauge field, the statistical transformation is generalized in higher dimensions by adopting tensor version of Chern-Simons field theory for membranes instead of particles. The mathematics of linking and phase interaction mediated by tensor gauge field in higher dimensions have already been formulated in Refs.[85, 76, 86, 88] [see Appendix D]. Based on the results, we discuss the statistical transformation and effective field theory for the A-class topological insulator. We will see that A-class topological insulator can be regarded as a superfluid state of composite membranes in the same way as the fractional quantum Hall effect is considered as a superfluid state of composite bosons.

Dim.	0	1	2	\cdots	p	$p+1$	\cdots	$2p+1$	$2p+2$
M_p	\circ	\circ	\circ	\cdots	\circ				

Table 3: We place a static p -brane in the space-time with dimension $2p+3$.

6.2 Tensor flux attachment

The flux attachment is achieved by applying the singular gauge transformation [89, 90, 45]. We first generalize this procedure in higher dimensions. Suppose p -brane occupying the dimensions from x^0 to x^p in $D = 2p+3$ [Table 3]. (Here, we render p as a non-negative integer not only an even integer.) From the remaining $(p+2)$ dimension $(x^{p+1}, \dots, x^{2p+2})$ p -brane is regarded as a point-particle. We apply the flux attachment to such a “point-particle” in $(p+2)$ -dimensional space. Technically, the gauge field associated with the flux readily be obtained by a “dimensional reduction” of the tensor monopole gauge field (157). On the equator of S^{p+2} ($x_{p+3} = 0$), the tensor monopole gauge field (157) is reduced to

$$A_{\mu_1 \mu_2 \cdots \mu_{p+1}} = -\Phi_p \frac{1}{\mathcal{A}(S^{p+1})} \frac{1}{r^{p+2}} \epsilon_{\mu_1 \mu_2 \cdots \mu_{p+2}} x^{\mu_{p+2}}, \quad (175)$$

where $\mu_1, \mu_2, \dots, \mu_{p+2} = p+1, p+2, \dots, 2p+2$ and $r^2 = \sum_{\mu=p+1}^{2p+2} x^\mu x^\mu$. For instance, we have

$$\begin{aligned} p=0 & : A_\mu = -\frac{\Phi_0}{2\pi r^2} \epsilon_{\mu\nu} x_\nu, \\ p=1 & : A_{\mu\nu} = -\frac{\Phi_1}{4\pi r^3} \epsilon_{\mu\nu\rho} x_\rho, \\ p=2 & : A_{\mu\nu\rho} = -\frac{\Phi_2}{2\pi^2 r^4} \epsilon_{\mu\nu\rho\sigma} x_\sigma. \end{aligned} \quad (176)$$

They are regarded as the tensor gauge field on the $(p+2)$ D plane [Fig.3]. With use of the Green

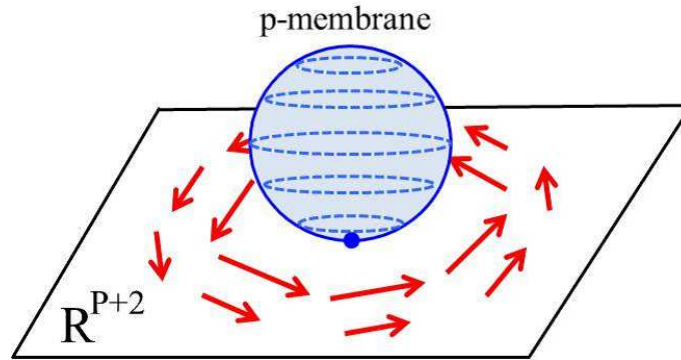


Figure 3: Flux is attached to membrane and yields the tensor gauge field around the membrane.

function in $(p+2)$ D space²³, (175) can be represented as

$$A_{\mu_1\mu_2\cdots\mu_{p+1}} = -\Phi_p \epsilon_{\mu_1\mu_2\cdots\mu_{p+2}} \partial_{\mu_{p+2}} G_{(p+2)}, \quad (179)$$

which take the form of “pure gauge”:

$$A_{\mu_1\mu_2\cdots\mu_{p+1}} = \frac{1}{p!} \partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_{p+1}]}, \quad (180)$$

where $\Lambda_{\mu_1\mu_2\cdots\mu_{p+1}}$ is formally expressed as

$$\Lambda_{\mu_1\mu_2\cdots\mu_p} = (-1)^{p+1} \Phi_p \epsilon_{\mu_1\mu_2\cdots\mu_{p+2}} \partial_{\mu_{p+1}} \frac{1}{\partial^2} \partial_{\mu_{p+2}} G_{(p+2)}. \quad (181)$$

The corresponding field strength

$$F_{\mu_1\mu_2\cdots\mu_{p+2}} = \frac{1}{(p+1)!} \partial_{[\mu_1} A_{\mu_2\mu_3\cdots\mu_{p+2}]} \quad (182)$$

is evaluated as

$$F_{\mu_1\mu_2\cdots\mu_{p+2}} = \epsilon_{\mu_1\mu_2\cdots\mu_{p+2}} B(x) \quad (183)$$

where B represents the flux-like magnetic field:

$$B(x) = \Phi_p \cdot \delta^{p+2}(x). \quad (184)$$

Φ_p stands for the strength of the flux. When a p -brane with charge e_p moves around the flux, the p -brane acquires the phase:

$$e^{ie_p \oint_{S^1 \times M_p} A} = e^{ie_p \int_{D_2 \times M_p} B} = e^{ie_p \Phi_p}, \quad (185)$$

where M_p denotes the configuration of p -brane. The phase should be 1:

$$e^{ie_p \Phi_p} = e^{2\pi i n}, \quad (186)$$

and then Φ_p is quantized as

$$\Phi_p = \frac{2\pi}{e_p} n, \quad (187)$$

²³ $G_{(d)}$ denotes Green function for the d -D Laplace equation:

$$\partial^2 G_{(d)}(x-y) = \delta^d(x-y), \quad (177)$$

where $\partial^2 = \sum_{\mu=1}^d \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu}$. Explicitly, the Green functions are given by

$$\begin{aligned} d=1, \quad & : \quad G_{(1)} = \frac{1}{2}|x| & \rightarrow & \quad \partial_x G_{(1)} = \pm \frac{1}{2} \text{sgn}(x), \\ d=2 \quad & : \quad G_{(2)} = \frac{1}{\mathcal{A}(S^1)} \ln r & \rightarrow & \quad \partial_\mu G_{(2)} = \frac{1}{\mathcal{A}(S^1)} \frac{1}{r^2} x_\mu, \\ d \geq 3 \quad & : \quad G_{(d)} = -\frac{1}{(d-2)\mathcal{A}(S^{d-1})} \frac{1}{r^{d-2}} & \rightarrow & \quad \partial_\mu G_{(d)} = \frac{1}{\mathcal{A}(S^{d-1})} \frac{1}{r^d} x_\mu. \end{aligned} \quad (178)$$

with integer n . Hence, the minimum unit of flux is given by²⁴

$$\hat{\Phi}_p = \frac{2\pi}{e_p}. \quad (190)$$

Let us consider a (composite) p -brane that carries κ fluxes:

$$Q_p = \kappa \hat{\Phi}_p, \quad (191)$$

where Q_p denotes the p -brane charge. In the $(D - p - 1)$ -dimensional space perpendicular to p -brane, (191) can locally be rewritten as

$$\rho_{\text{eff}}(x_\perp) = \frac{1}{e_p} B_{\text{eff}}(x_\perp), \quad (192)$$

where

$$e_p \rho_{\text{eff}}(x_\perp) = Q_p \delta^{(p+2)}(x_\perp), \quad B_{\text{eff}}(x_\perp) = \kappa \hat{\Phi}_p \delta^{(p+2)}(x_\perp), \quad (193)$$

with $x_\perp^\mu = (x^{p+1}, x^{p+2}, \dots, x^{D-1})$. Furthermore, one may readily derive (192) by integrating

$$\rho(x) = \frac{1}{e_p} B(x), \quad (194)$$

over the space parallel to p -brane, $x_\parallel = (x^1, x^2, \dots, x^p)$, with use of

$$e_p \rho_{\text{eff}}(x_\perp) = e_p \int d^p x_\parallel \rho(x), \quad B_{\text{eff}}(x_\perp) = \int d^p x_\parallel B(x). \quad (195)$$

Here, $J_{\mu_1 \mu_2 \dots \mu_{p+1}}(x)$ denotes the p -brane current²⁵ and $\rho(x)$ and $B(x)$ are given by

$$\rho(x) = J^{012 \dots p-1}(x), \quad B(x) = F_{p+1, p+2, \dots, 2p+2}(x). \quad (199)$$

²⁴ (190) is consistent with the result of the charge quantization of monopole:

$$e_p e_{D-p-4} = 2\pi n. \quad (188)$$

This manifests Dirac quantization condition between p and $(D - p - 4)$ -branes. Since non-overlapping p and $(D - p - 4)$ branes occupy $D - 3$ spacial dimensions, from the co-dimension 3, the p and $(D - p - 4)$ -branes are regarded as point-like objects, and so we can apply the ordinary Dirac quantization condition to the charges of p and $(D - p - 4)$ branes (188) in the same way as electron and monopole in 3D. Consequently, the minimum unit of the $(D - p - 4)$ -brane charge is derived as

$$\Delta e_{D-p-4} = \frac{2\pi}{e_p}, \quad (189)$$

which is consistent with (187).

²⁵The explicit form of the membrane current is given as follows. We place p -brane in the dimensions, (x^1, x^2, \dots, x^p) , and we parametrize the coordinates of membrane as

$$\begin{aligned} x_\parallel^\mu &= X^\mu(\sigma), \quad (\mu = 1, 2, \dots, p), \\ x_\perp^\mu &= 0, \quad (\mu = p+1, \dots, D-1) \end{aligned} \quad (196)$$

where $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^p)$ denotes the intrinsic coordinates of the p -brane. Non-vanishing component of p -brane current is given by

$$J^{012 \dots p}(x) = \int d^p \sigma \det\left(\frac{\partial X}{\partial \sigma}\right) \delta^{(D)}(x - X(\sigma)) = \delta^{(D-p-1)}(x_\perp) \int d^p \sigma \det\left(\frac{\partial X}{\partial \sigma}\right) \delta^{(p)}(x_\parallel - X(\sigma)), \quad (197)$$

Consequently, one may find the covariant expression for (194):

$$J_{\mu_1\mu_2\cdots\mu_{p+1}}(x) = \frac{1}{(p+2)!} \frac{1}{e_p} \epsilon_{\mu_1\mu_2\cdots\mu_{2p+3}} F^{\mu_{p+2}\cdots\mu_{2p+3}}(x). \quad (200)$$

This realizes the tensor flux attachment to p -brane in $(2p+3)$ D space(-time), and is a natural generalization of the flux attachment in 3D space(-time):

$$J_\mu = \frac{1}{2e_0} \epsilon_{\mu\nu\rho} F^{\nu\rho}. \quad (201)$$

6.3 $(2k-2)$ -brane as the $SO(2k+1)$ skyrmion

In the realization of the fractional statistics for the $SO(3)$ nonlinear model in (2+1)D [82, 83], the statistical gauge field is coupled to the $SO(3)$ skyrmion topological current. The underlying mathematics of the $SO(3)$ skyrmion is given by the 1st Hopf map [82], where the target space S^2 (7) corresponds to the field manifold of skyrmion. Since both of $SO(3)$ non-linear sigma model and the Haldane's two-sphere are based on the 1st Hopf map, the mathematical structure of the $SO(3)$ non-linear sigma model is quite similar to that of the Haldane's two-sphere [28]; The *internal* field manifold of the $SO(3)$ skyrmion is S^2 and the “hidden” local symmetry is $U(1)$, while in the Haldane's two-sphere the *external* space is S^2 and the gauge symmetry is $U(1)$. Thus interestingly, we can “interchange” the set-ups for the $SO(3)$ non-linear sigma model and the Haldane's two-sphere by exchanging external and internal spaces. Subsequently, the authors in [76, 85, 86, 77] adopted the 2nd Hopf map (and the 3rd Hopf map also) to construct the $SO(5)$ non-linear sigma model for 2-brane on a four-sphere. We further apply this idea to construct the non-linear sigma model for membrane of higher dimensional quantum Hall effect. Since $2k$ D quantum Hall effect accommodates the “internal” $(2k-2)$ -brane on the external space S^{2k} , the corresponding non-linear sigma model is the $SO(2k+1)$ non-linear sigma model realizing a skyrmion solution spatially extended over S^{2k-2} with S^{2k} internal space. The internal space coordinates of the $SO(2k+1)$ skyrmion are given by

$$n = \sum_{a=1}^{2k+1} n_a \gamma_a \quad (202)$$

where n is subject to the condition of S^{2k} :

$$n^2 = \sum_{a=1}^{2k+1} n_a n_a = 1. \quad (203)$$

Following to the Derrick's theorem, there does not exist static soliton solutions in the scalar field theory whose Lagrangian only consists of the second order kinetic term, $\text{tr}(\partial^\mu n)^\dagger (\partial_\mu n)$, and self-interaction potential in the space-time whose dimension is larger than 2. However, there are at

and the total charge Q_p is

$$Q_p = e_p \int d^{D-1}x J^{012\cdots D-1}(x) = e_p \int d^p\sigma \det\left(\frac{\partial X}{\partial \sigma}\right) = e_p \cdot V_p, \quad (198)$$

with V_p the volume of the p -brane, $V_p \equiv \int d^p\sigma \det\left(\frac{\partial X}{\partial \sigma}\right)$.

least two ways to evade the Derrick's theorem. One is to include an extra interaction term to stabilize the soliton configuration, and the other is to adopt a higher derivative kinetic term [91]. Here we just suppose that the skyrmion configuration is stabilized by taking some method to evade the theorem.

The $(2k - 2)$ -brane charge is given by the $SO(2k + 1)$ skyrmion topological number,

$$\pi_{2k}(S^{2k}) \simeq \mathbb{Z}. \quad (204)$$

In $(4k - 1)$ D space-time, the $SO(2k + 1)$ skyrmion or $(2k - 2)$ -brane current is constructed as

$$J^{\mu_1 \mu_2 \dots \mu_{2k-1}} = \frac{1}{(2k)!} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon^{\mu_1 \mu_2 \dots \mu_{4k-1}} n^{a_1} \partial_{\mu_{2k}} n^{a_2} \partial_{\mu_{2k+1}} n^{a_3} \dots \partial_{\mu_{4k-1}} n^{a_{2k+1}}, \quad (205)$$

where $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$ ($\mu = 0, 1, 2, \dots, 4k - 2$). In the differential form, (205) is simply represented as²⁶

$$J_{2k} = \frac{1}{(2i)^k} \text{tr}(n(dn)^{2k}), \quad (208)$$

where $\text{tr}(\gamma_{\mu_{2k+1}} \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_{2k}}) = (2i)^k \epsilon_{\mu_1 \mu_2 \mu_3 \dots \mu_{2k+1}}$ was used. The topological number of the $SO(2k + 1)$ skyrmion is given by

$$N = \frac{1}{\mathcal{A}(S^{2k})} \int_{S^{2k}} J_{2k}. \quad (209)$$

6.4 Flux cancellation and tensor Chern-Simons theory

Topological features of the fractional quantum Hall effect are nicely captured by the Chern-Simons effective field theory[43, 44]. The Chern-Simons field is introduced to cancel the external magnetic field, and the odd number Chern-Simons fluxes attachment transmutes electron to composite boson. In 2D, both of the external magnetic field and the Chern-Simons field are $U(1)$, and then the relation for flux cancellation is rather trivial

$$2\text{D} : A - C_1 = 0, \quad (210)$$

Meanwhile in higher dimensions, we have to deal with the non-abelian external field and membranes. One may wonder how we can incorporate these two objects to generalize the flux cancellation. The non-abelian and tensor monopole relation (149) gives a crucial hint. We demonstrated that the non-abelian gauge field is “equivalent” to the $U(1)$ tensor gauge field. This suggests that the cancellation of the external *non-abelian* gauge field by *abelian* gauge (tensor) field is

²⁶ $(D - p - 1)$ form current J is introduced as

$$(*J)_{p+1} = \frac{1}{(p+1)!} J_{\mu_1 \mu_2 \dots \mu_{p+1}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{p+1}}, \quad (206)$$

and so

$$J_{D-p-1} = \frac{1}{(p+1)!(D-p-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_D} J_{\mu_1 \mu_2 \dots \mu_{p+1}} dx_{\mu_{p+2}} dx_{\mu_{p+3}} \dots dx_{\mu_D}. \quad (207)$$

The original space-time D.	3	5	7	9	11	13	15	17	19	...
The emergent space-time D.	3	7	11	15	19	23	27	31	35	...

Table 4: The effective field theory of A-class topological insulators in $(2k + 1)$ D space-time is given by the tensor Chern-Simons theory of $(2k - 2)$ -branes in $(4k - 1)$ D space-time.

possible. We thus consider the $U(1)$ Chern-Simons tensor flux attachment to membrane and the flux cancellation condition is generalized in higher dimensions as

$$2kD : \text{tr}(L_{CS}^{(2k-1)}[A]) - C_{2k-1} = 0. \quad (211)$$

In low dimensions, (211) yields

$$\begin{aligned} 4D : \text{tr}(AdA + \frac{2}{3}iA^3) - C_3 &= 0, \\ 6D : \text{tr}(A(dA)^2 + \frac{3}{2}iA^3dA - \frac{3}{5}A^5) - C_5 &= 0. \end{aligned} \quad (212)$$

Since the membranes are the fundamental object in A-class topological insulator, it is natural to reformulate the theory by using the membrane degrees of freedom. We propose the Chern-Simons tensor field theory as the effective field theory for A-class topological insulators ²⁷

$$S = e_p \int_{2p+3} C_{p+1} J_{p+2} + \frac{\kappa}{2} \int_{2p+3} C_{p+1} G_{p+2}, \quad (213)$$

where J_{p+2} is given by the membrane $p + 2$ form current (208) and

$$G_{p+2} = dC_{p+1}. \quad (214)$$

The action (213) is equivalent to the one used in the analysis of linking of membrane currents [86]. For $p = 2k - 2$, the Chern-Simons coupling is given by

$$\kappa = \frac{1}{\hat{\Phi}_p} \nu_{2k} = \frac{e_p}{2\pi} \frac{1}{m^k}, \quad (215)$$

where $\hat{\Phi}_p$ denotes the unit-flux (190) and ν_{2k} stands for the filling factor of $(2k - 2)$ -brane (113). Notice that while the original space-time dimension is $(2k + 1)$ that the tensor Chern-Simons theory is defined in $(4k - 1)$ D space(-time) (for $p = 2k - 2$) [Table 4]. Thus, the tensor Chern-Simons theory is formulated in the enlarged space as consistent with the observation in Sec.4.2. Since there does not exit the kinetic term in the action, C_{p+1} is not a dynamical field but an

²⁷In [88], the authors adopted the ordinary $(6+1)$ D $U(1)$ Chern-Simons theory as an effective field theory for 4D quantum Hall effect, which describe 0-branes rather than membranes.

auxiliary field determined by the equations of motion ²⁸

$$J_{p+2} = -\frac{\kappa}{e_p} G_{p+2}. \quad (218)$$

In the space-time components, (218) can be written as

$$J^{i_1 i_2 \dots i_p 0} = (-1)^{p+3} \frac{\kappa}{e_p} B^{i_1 i_2 \dots i_p}, \quad (219a)$$

$$J^{i_1 i_2 \dots i_{p+2}} = (-1)^{p+2} \frac{1}{(p+2)!} \frac{\kappa}{e_p} \epsilon^{i_1 i_2 \dots i_{2p+2}} E_{i_{p+2} \dots i_{2p+2}}, \quad (219b)$$

where $\epsilon^{i_1 i_2 \dots i_{2p+2}} \equiv \epsilon^{i_1 i_2 \dots i_{2p+2} 0}$ and

$$\begin{aligned} E_{i_1 i_2 \dots i_{p+2}} &\equiv G_{0 i_1 i_2 \dots i_{p+2}} = \frac{1}{(p+2)!} \partial_{[0} C_{i_1 i_2 \dots i_{p+2}]}, \\ B^{i_1 i_2 \dots i_p} &= \frac{1}{(p+2)!} \epsilon^{i_1 i_2 \dots i_{2p+2}} G_{i_{p+1} \dots i_{2p+2}}. \end{aligned} \quad (220)$$

(219a) indeed realizes the generalization the flux attachment for membrane (200) and suggests that the membrane with unit charge e_p carries m^k fluxes in unit of $\hat{\Phi}_p$. Meanwhile (219b) gives a generalization of the Hall effect. From the antisymmetric property of the epsilon tensor, we have

$$E_{i_1 i_2 \dots i_{p+1}} J^{i_1 i_2 \dots i_{p+1}} = (-1)^{p+1} E_{i_1 i_2 \dots i_{p+1}} J^{i_1 i_2 \dots i_{p+1}}. \quad (221)$$

Since the membranes are always an even dimensional object ($p = 2k-2$), the Hall effect necessarily holds:

$$E_{i_1 i_2 \dots i_{p+1}} J^{i_1 i_2 \dots i_{p+1}} = 0. \quad (222)$$

Meanwhile if p was odd, the Hall effect would *not* necessarily hold.

6.5 Composite membrane and fractional charge

Integration of the Chern-Simons field in the tensor Chern-Simons action gives a generalized Gauss-Hopf linking between two membrane world volumes, which can alternatively be understood as the winding number from the two higher dimensional “tori” to a higher dimensional sphere [see [86] or Appendix D]:

$$(S^{2k-2} \times S^1) \times (S^{2k-2} \times S^1) \rightarrow S^{4k-1}. \quad (223)$$

From (223), it is obvious that the non-trivial winding exists for arbitrary k , and so does the linking. Even though the membrane statistics is related to the linking, it does not necessarily mean that

²⁸In component representation, (213) and (218) are respectively expressed as

$$S = \frac{1}{(p+1)!} \int d^{2p+3} x \left(-e_p J_{\mu_1 \mu_2 \dots \mu_{p+1}} C^{\mu_1 \mu_2 \dots \mu_{p+1}} + \frac{\kappa}{2(p+2)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{2p+3}} C_{\mu_1 \mu_2 \dots \mu_{p+1}} G_{\mu_{p+2} \mu_{p+3} \dots \mu_{2p+3}} \right), \quad (216)$$

$$J^{\mu_1 \mu_2 \dots \mu_{p+1}} = -\frac{1}{(p+2)!} \frac{\kappa}{e_p} \epsilon^{\mu_1 \mu_2 \dots \mu_{2p+3}} G_{\mu_{p+2} \mu_{p+3} \dots \mu_{2p+3}}. \quad (217)$$

membranes obey the fractional statistics. For instance in quantum Hall effect, for quasi-excitation to be anyonic, the fractional charge is essential [79]. Similarly, for statistical transmutation from electron to (composite) boson, the odd number flux attachment is crucial, and hence quantum Hall liquid at the magic filling $\nu = 1/m$ for odd m is considered as a superfluid state of the composite bosons [45, 46, 43].

First, we consider the composite boson counterpart in A-class topological insulators. At $\nu = 1/m^k$, m^k fluxes are attached to the membrane and the membrane becomes to a composite object of membrane and the fluxes. The original statistics of the membrane is fermionic since at $\nu = 1$ membrane corresponds to “quarks” with color degrees of freedom. The statistics of the composite membrane is derived by evaluating the phase interaction between two composite membranes. Under the interchange, the membranes acquires the following statistical phase

$$e^{i\frac{1}{2}e_{2k-2} \oint A} = e^{i\pi m^k} = -1, \quad (224)$$

where we used $\oint A = m^k \Phi_{2k-2}$ ($\Phi_{2k-2} = \frac{2\pi}{e_{2k-2}}$) and m is odd so is m^k . Since the membrane acquires the extra minus sign under the interchange of membranes, by the flux attachment the membrane transmutes its statistics from fermion to boson, and obeys the Bose statistics. Notice that such transmutation is only possible when the inverse of the magic filling $\nu_{2k} = 1/m^k$ is odd. In the same way as the fractional quantum Hall effect at $\nu = 1/m$ is considered as a condensation of composite bosons, A-class topological insulator at $\nu = 1/m^k$ can be interpreted as a superfluid state of composite membranes. Next let us discuss the statistics of membrane excitation. We first need to specify the membrane charge. When the monopole charge is $I/2$, the number of states on S^{2k} is given by

$$\sim I^k, \quad (225)$$

and for the filling $\nu = 1$ the $(2k-2)$ -brane with unit charge e_{2k-2} , occupies each state. When the monopole charge change as $I' = mI$, the number of states becomes to

$$I'^k = m^k I^k. \quad (226)$$

In other words, each state occupied by membrane is split to m^k states, and so does the membrane charge. Hence at $\nu_{2k} = \frac{1}{m^k}$, the fractional charge of $(2k-2)$ -brane is given by²⁹:

$$e'_{2k-2} = \frac{1}{I'^k} I^k e_{2k-2} = \frac{1}{m^k} e_{2k-2}. \quad (230)$$

²⁹ (230) can also be derived from the perspective of 0-branes. When the monopole charge is $I/2$, the $(2k-2)$ -brane is made of $I^{\frac{1}{2}k(k-1)}$ 0-branes, and then $(2k-2)$ -brane charge is expressed by

$$e_{2k-2} = \kappa(k) \cdot I^{\frac{1}{2}k(k-1)} e_0, \quad (227)$$

where $\kappa(k)$ is a coefficient of dimension of $(\text{mass})^{2k-2}$. At $I' = mI$, the 0-brane charge becomes to

$$e'_0 = \frac{1}{m^{\frac{1}{2}k(k+1)}} e_0, \quad (228)$$

and so the $(2k-2)$ -brane charge reads as

$$e'_{2k-2} = \kappa(k) \cdot I'^{\frac{1}{2}k(k-1)} e'_0 = \kappa(k) \cdot \frac{1}{m^{\frac{1}{2}k(k+1)}} I'^{\frac{1}{2}k(k-1)} e_0 = \frac{1}{m^k} e_{2k-2}. \quad (229)$$

Since the $(2k - 2)$ -brane excitation is induced by the flux penetration, $(2k - 2)$ -brane excitation is a “composite” of the fractional charge e'_{2k-2} and the unit flux $\hat{\Phi}_p = \frac{2\pi}{e_{2k-2}}$. Therefore, the geometrical phase which a fractionally charged $(2k - 2)$ -brane acquires by the round trip around another $(2k - 2)$ -brane excitation is given by

$$e^{ie'_{2k-2} \oint A} = e^{ie'_{2k-2} \hat{\Phi}_p} = e^{2\pi i \frac{e'_{2k-2}}{e_{2k-2}}} = e^{\frac{2\pi}{m^k} i}. \quad (231)$$

We thus have shown that the statistical phase of membrane excitation is $2\pi\nu_{2k}$ and hence membrane excitations are anyonic.

6.6 Dimensional hierarchy and analogies to string theory

Analogies between the A-class topological insulator and the string theory will be transparent in analyses of membrane properties. According to the Haldane-Halperin picture [28, 80], quasi-particles condense on the parent quantum Hall liquid to generate a new incompressible liquid and the filling factor exhibits a hierarchical structure called Haldane-Halperin hierarchy. Similarly in A-class topological insulator, membrane excitations are expected to condense to form a new incompressible liquid and the filling factor will exhibit a generalized Haldane-Halperin like hierarchy:

$$\nu_{2k} = \frac{1}{m^k \pm \frac{1}{(2p_1)^k \pm \frac{1}{(2p_2)^k \pm \dots}}}, \quad (232)$$

where each of p_1, p_2, \dots denotes a natural number. Apart from the Halperin-Haldane hierarchy, the membranes exhibit a unique type of condensation – the dimensional hierarchy [31, 32], which reflects the special dimensional pattern of A-class topological insulator. From (81), one may find that there is a relation between $2k$ and $(2k - 2)$ D lowest Landau level degeneracies:

$$D_{LLL}(k, I) \sim I^k D_{LLL}(k - 1, I), \quad (233)$$

and then

$$D_{LLL}(k, I) \sim I^k \cdot I^{k-1} \cdot I^{k-2} \dots I^2 \cdot I = I^{\frac{1}{2}k(k+1)}. \quad (234)$$

Eq.(234) suggests a hierarchical structure in dimensions. This feature can intuitively be understood by the following simple explanations. Each of the $SO(2k)$ monopole fluxes on S^{2k} occupies an area $\ell_B^{2k} = (\alpha r)^k = (2r^2/I)^k$, and the number of fluxes on S^{2k} is given by $\sim r^{2k}/\ell_B^{2k} \sim I^k$. Since the $SO(2k)$ non-abelian flux is equivalent to $(2k - 2)$ -brane, one may say $(2k - 2)$ -brane occupies the same area ℓ_B^{2k} and $\sim I^k$ is the number of $(2k - 2)$ -branes. Similarly, on S^{2k-2} , there are $(2k - 4)$ -branes each of which occupies the area ℓ_B^{2k-2} , and the total number of $(2k - 4)$ -branes is $\sim I^{k-1}$. By repeating this iteration from $2k$ D to the lowest dimension 2D, we obtain the formula (233). The corresponding filling factor (for 0-brane) is given by

$$\nu = \frac{1}{m} \frac{1}{m^2} \frac{1}{m^3} \dots \frac{1}{m^{k-1}} \frac{1}{m^k} = \frac{1}{m^{\frac{1}{2}m(m+1)}}. \quad (235)$$

Similar to the Haldane-Halperin hierarchy, such hierarchical structure may imply a particular condensation of membranes which ranges in dimensions. Inversely, one may see the formula from

low dimension and find a physical interpretation of the hierarchy; low dimensional membranes gather to form a higher dimensional incompressible liquid of membranes [Fig.4]. Most general total filling factor will be give by the combination of (232) and (235),

$$\nu = \nu_2 \nu_4 \cdots \nu_{2k} = \frac{1}{m \pm \frac{1}{2p_1 \pm \frac{1}{2p_2 \pm \cdots}}} \cdot \frac{1}{m^2 \pm \frac{1}{(2p_1)^2 \pm \frac{1}{(2p_2)^2 \pm \cdots}}} \cdots \frac{1}{m^k \pm \frac{1}{(2p_1)^k \pm \frac{1}{(2p_2)^k \pm \cdots}}}. \quad (236)$$

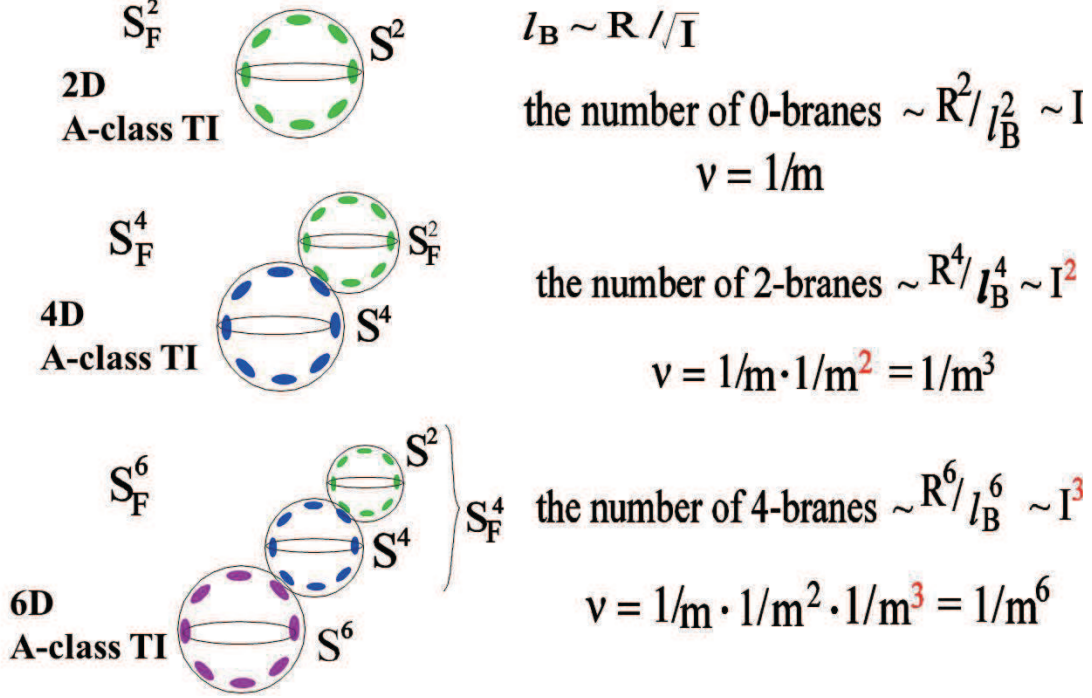


Figure 4: Low dimensional membranes condense to to form a higher dimensional membrane. Since the membrane itself describes fuzzy sphere or A-class topological insulator, one may alternatively state this phenomena as the dimensional hierarchy of A-class topological insulator.

Since $\nu_2, \nu_4, \cdots, \nu_{2k}$ are equally treated in (236), one can arbitrarily interchange ν s in different dimensions, which suggests a “democratic” property of A-class topological insulator. The filling factor denotes the membrane density and the interchangeability of the filling factors implies a equivalence of membranes in different dimensions. This may immediately remind the brane democracy of string theory; any D-brane can be a starting point to construct another D-brane in different dimensions [94]. Thus, the dimensional hierarchy – the membranes condense to make an incompressible liquid – can be regarded as a physical realization of the brane democracy. The index theorem also suggests the close relations between the A-class topological insular and the string theory. The index theorem tell that the lowest Landau level degeneracy, $D_{LLL}(k-1, I)$, is equal to the $(k-1)$ th Chern-number, $c_{k-1}(I+1)$. This equality means that the $(k-1)$ th Chern number is identical to the $(2k-2)$ -brane charge, since the number of 0-branes is given by the lowest Landau level degeneracy. Analogous phenomena have been reported in the context of Myers effect of string theory [34] where low dimensional D-branes on higher dimensional D-brane

are regarded as magnetic fluxes of monopole. In particular, Kimura found that the number of D0-branes that constitute a spherical $D(2k-2)$ -brane is given by the $(k-1)$ th Chern-number of non-abelian monopole [35]. The fact that the membrane charge is equal to the lowest Landau level degeneracy – the number of the fundamental elements of the lowest Landau level, implies that membranes themselves should be identified with the fundamental elements of the space(-time). This observation again reminds the idea of the matrix theory [92, 93] in which the D0 ($D-1$) branes constitute the space(-time) and the spacial coordinates are represented by matrices. It is quite interesting that the ideas of the string theory can be understood in the context of topological insulators.

7 Summary and Discussions

We discussed physical realization of the quantum Nambu geometry in the context of A-class topological insulator. As the higher dimensional dimensional fuzzy sphere has two different formulations, A-class topological insulator has two physically different realizations, one of which is the non-abelian monopole realization and the other is the tensor monopole realization. We established the connection between these two kinds of monopole through the Chern-Simons term. Based on the non-abelian and tensor connection, we generalized the flux attachment procedure in A-class topological insulator to construct the Chern-Simons tensor effective field theory. We also showed the exotic concepts in 2D quantum Hall effect can be naturally generalized to A-class topological insulators.

For convenience of readers, we summarize the main achievements of the present work. In arbitrary even dimension we established

- Equality between monopole charge and the lowest Landau level degeneracy via the index theorem [Sec.4.3, 4.4]
- Connection between the non-abelian and tensor monopoles [Sec.5.2]

Based on the above observations, we derived

- Explicit form of the tensor monopole gauge fields from the non-abelian monopole gauge fields [Sec.5.3]
- Non-commutative coordinates of quantum Nambu geometry via angular momentum construction [Sec.5.4]

Subsequently, we discussed their physical consequences in the context of A-class topological insulators:

- Tensor flux attachment to membrane and its statistical phase [Sec.6.2]
- Higher D generalization of flux cancellation and Chern-Simons tensor field theory [Sec.6.4]
- Fractional charge and anyonic statistics for membrane [Sec.6.5]

Though the original space-time of A-class topological insulators is arbitrary odd dimensional space-time $(2k + 1)$, the effective Chern-Simons tensor field theory lives in $(4k - 1)$ dimensional space-time, not in arbitrary odd dimensions. Since the Chern-Simons theories are defined in arbitrary odd dimensions, one may wonder why we “skipped” the Chern-Simons theories in other odd dimensions, $5, 9, 13, \dots$. We will address the issue in the forth coming paper. The edge theory and accompanied Callan Harvey mechanism based on the Chern-Simons tensor field theory may also be interesting.

The quantum Nambu bracket has attracted a lot of attentions in recent years since it is expected to provide an appropriate description for M-brane boundstate [95] and plays a vital role in Bagger-Lambert-Gustavsson theory of multiple M-branes [96, 97, 98]. Non-associative geometry associated with the quantum Nambu bracket has also been vigorously studied [99, 100]. Zhang, a pioneer of topological insulator and higher D. quantum Hall effect, noted that the study of condensed matter physics may provide an alternative approach to understand exotic ideas in mathematical and particle physics [101]. We thus enforced his observation by demonstrating quantum Nambu geometry in A-class topological insulators inspired by the recent works [14, 15]. We hope the present work will further deepen the understanding of non-commutative geometry and string theory as well as topological insulators.

Acknowledgments

The author is grateful to Shoucheng Zhang for his interests in this work and accepting the author as a visiting scholar in Stanford University. Discussions and email correspondences with Koji Hashimoto, Shoichi Kawamoto, Taro Kimura, Yusuke Kimura, Mohammad M. Sheikh-Jabbari, Richard J. Szabo and Tomohisa Takimi were helpful to develop this work. I appreciate the valuable communications with them. Finally, I would like to thank Emma Akiyama and Takahide Akiyama for their warm encouragements and supports. Presentations and discussions during the workshops “Exotic Space-Time Geometry and Its Application” held at Riken in Wako, Saitama, Japan, Feb. 23, 2013 and “Noncommutative Field Theory and Gravity” held in Corfu, Greece, Sept. 8-15, 2013, were useful to complete the present work. This work was partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan (Grant No.23740212), Overseas Dispatching Program 2013 of National College of Technology, and The Emma Project for Art and Culture.

A Fully symmetric representations of $SO(2k + 1)$ and $SO(2k)$

In the $SO(2k + 1)$ fully symmetric representation $\overbrace{[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}]}^k$, the gamma matrices satisfy

$$\sum_{a=1}^{2k+1} G_a G_a = I(I + 2k) \quad (237)$$

and

$$[G_{a_1}, G_{a_2}, \dots, G_{a_{2k}}] = i^k C'(k, I) \cdot \epsilon_{a_1 a_2 \dots a_{2k+1}} G_{a_{2k+1}}, \quad (238)$$

where $C'(k, I)$ is given by

$$C'(k, I) \equiv \frac{(2k)!!(I + 2k - 2)!!}{I!!}. \quad (239)$$

The $SO(2k + 1)$ generators are constructed as

$$G_{ab} = -i \frac{1}{4} [G_a, G_b]. \quad (240)$$

G_a and G_{ab} satisfy the closed algebra:

$$\begin{aligned} [G_a, G_b] &= 4i G_{ab}, \\ [G_a, G_{bc}] &= -i(\delta_{ab} G_c - \delta_{ac} G_b) \\ [G_{ab}, G_{cd}] &= i(\delta_{ac} G_{bd} - \delta_{ad} G_{bc} + \delta_{bd} G_{ac} - \delta_{bc} G_{ad}), \end{aligned} \quad (241)$$

which is identical to the $SO(2k + 2)$ algebra. X_a and X_{ab} operators of S_F^{2n} are constructed as

$$\begin{aligned} X_a &= \frac{\alpha}{2} G_a, \\ X_{ab} &= \alpha G_{ab}, \end{aligned} \quad (242)$$

with $\alpha = 2r/I$ (3). For $I = 1$, G_a and G_{ab} are reduced to the fundamental representation, Γ_a (44) and $\Sigma_{ab} = -i \frac{1}{4} [\Gamma_a, \Gamma_b]$.

The $SO(2k)$ group has two Weyl representations, $\Sigma_{\mu\nu}^+$ and $\Sigma_{\mu\nu}^-$ ($\mu, \nu = 1, 2, \dots, 2k$). For the fundamental representation $I = 1$, the $SO(2k)$ Weyl generators satisfy

$$\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \dots \mu_{2k}} \Sigma_{\mu_3 \mu_4}^\pm \dots \Sigma_{\mu_{2k-1} \mu_{2k}}^\pm = \pm \frac{(2k-2)!}{2^{k-2}} \Sigma_{\mu_1 \mu_2}^\pm, \quad (243a)$$

$$\text{tr}(\Sigma_{\mu_1 \mu_2}^\pm \Sigma_{\mu_2 \mu_3}^\pm) = -2^{k-3} (2k-1) \delta_{\mu_1 \mu_3}. \quad (243b)$$

and for the fully symmetric representation $\left[\overbrace{\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}}^k \right]$,

$$\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \dots \mu_{2k}} \Sigma_{\mu_3 \mu_4}^\pm \dots \Sigma_{\mu_{2k-1} \mu_{2k}}^\pm = \pm \frac{1}{2^{k-2}} C'(k-1, I) \Sigma_{\mu_1 \mu_2}^\pm, \quad (244a)$$

$$\text{tr}(\Sigma_{\mu_1 \mu_2}^\pm \Sigma_{\mu_2 \mu_3}^\pm) = -\frac{1}{4} D_{LLL}(k-1, I) I(2k+I-2) \delta_{\mu_1 \mu_3}. \quad (244b)$$

Here, $D_{LLL}(k-1, I)$ denotes the dimension of the $SO(2k)$ fully symmetric representation equal to the dimension of the $SO(2k-1)$ fully symmetric representation (233). For the fundamental representation, $G_{\mu\nu}$ and $\Sigma_{\mu\nu}^\pm$ are related by (46), and for the generic fully symmetric representation $G_{\mu\nu}$ can be represented as a block diagonal form and $\Sigma_{\mu\nu}^\pm$ appear in the left-up and right-down blocks:

$$G_{\mu\nu} = \begin{pmatrix} \Sigma_{\mu\nu}^+ & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Sigma_{\mu\nu}^- \end{pmatrix}. \quad (245)$$

B Properties of quantum Nambu bracket

For $d = n + 1$ dimensional space, (32) can be written as

$$[X_{a_1}, X_{a_2}, \dots, X_{a_n}] = \epsilon_{a_1 a_2 \dots a_n a_{n+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_n} a_{n+1}} X_{b_{a_1}} X_{b_{a_2}} \dots X_{b_{a_n}}, \quad (246)$$

where $a_1, a_2, a_{n+1}, b_1, b_2, b_{n+1} = 1, 2, \dots, n + 1$. For instance,

$$[X_1, X_2, \dots, X_n] = \epsilon_{\mu_1 \mu_2 \dots \mu_n} X_{\mu_1} X_{\mu_2} \dots X_{\mu_n}, \quad (247)$$

where $\mu_1, \mu_2, \dots, \mu_n = 1, 2, \dots, n$. From the formula

$$\epsilon_{a_1 a_2 \dots a_n a_{n+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_n} a_{n+1}} = \det \begin{pmatrix} \delta_{a_1 b_{a_1}} & \delta_{a_1 b_{a_2}} & \dots & \delta_{a_1 b_{a_n}} \\ \delta_{a_2 b_{a_1}} & \delta_{a_2 b_{a_2}} & \dots & \delta_{a_2 b_{a_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{a_n b_{a_1}} & \delta_{a_n b_{a_2}} & \dots & \delta_{a_n b_{a_n}} \end{pmatrix} \equiv \det(\delta_{a_i b_{a_j}}) \quad (i, j = 1, 2, \dots, n), \quad (248)$$

(246) can be rewritten as

$$[X_{a_1}, X_{a_2}, \dots, X_{a_n}] = \det(\delta_{a_i b_{a_j}}) X_{b_{a_1}} X_{b_{a_2}} \dots X_{b_{a_n}}. \quad (249)$$

It is obvious that (246) can be represented as the commutator or the anti-commutator of the “sub-brackets”:

$$\begin{aligned} & [X_{a_1}, X_{a_2}, \dots, X_{a_n}] \\ &= \frac{1}{m!(n-m)!} \epsilon_{a_1 a_2 \dots a_n a_{n+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_n} a_{n+1}} [X_{b_{a_1}}, X_{b_{a_2}}, \dots, X_{b_{a_m}}] [X_{b_{a_{m+1}}}, \dots, X_{b_{a_n}}] \\ &= \frac{1}{2m!(n-m)!} \epsilon_{a_1 a_2 \dots a_n a_{n+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_n} a_{n+1}} [[X_{b_{a_1}}, X_{b_{a_2}}, \dots, X_{b_{a_m}}], [X_{b_{a_{m+1}}}, \dots, X_{b_{a_n}}]]_{(-1)^{m(n-m)}}, \\ & \quad (m \leq n) \end{aligned} \quad (250)$$

where $[\]_+ \equiv \{ \ }$ and $[\]_- \equiv [\]$. Thus, the n bracket has the hierarchical structure; the larger n bracket is decomposed to the of sub-brackets. In particular, for $n = 2k$, $2k$ bracket can be represented by the 2 brackets:

$$\begin{aligned} [X_{a_1}, X_{a_2}, \dots, X_{a_{2k}}] &= \frac{1}{2^k} \epsilon_{a_1 a_2 \dots a_{2k} a_{2k+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_{2k}} a_{2k+1}} [X_{b_{a_1}}, X_{b_{a_2}}] [X_{b_{a_3}}, X_{b_{a_4}}] \dots [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \\ &= \frac{1}{2^{2k-1}} \epsilon_{a_1 a_2 \dots a_{2k} a_{2k+1}} \epsilon_{b_{a_1} b_{a_2} \dots b_{a_{2k}} a_{2k+1}} \{ \dots \{ [X_{b_{a_1}}, X_{b_{a_2}}], [X_{b_{a_3}}, X_{b_{a_4}}] \}, [X_{b_{a_5}}, X_{b_{a_6}}] \} \dots \{ [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \} \end{aligned} \quad (251)$$

In particular,

$$\begin{aligned} [X_1, X_2, \dots, X_{2k}] &= \frac{1}{2^k} \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} [X_{\mu_1}, X_{\mu_2}] [X_{\mu_3}, X_{\mu_4}] \dots [X_{\mu_{2k-1}}, X_{\mu_{2k}}] \\ &= \frac{1}{2^{2k-1}} \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} \{ \dots \{ [X_{\mu_1}, X_{\mu_2}], [X_{\mu_3}, X_{\mu_4}] \}, \dots \{ [X_{\mu_{2k-1}}, X_{\mu_{2k}}] \}, \end{aligned} \quad (252)$$

with $\mu_1, \mu_2, \dots, \mu_{2k} = 1, 2, \dots, 2k$. For $k = 2, 3$, we have

$$\begin{aligned} [X_1, X_2, X_3, X_4] &= \frac{1}{8} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \{[X_{\mu_1}, X_{\mu_2}], [X_{\mu_3}, X_{\mu_4}]\} \\ &= \{[X_1, X_2], [X_3, X_4]\} - \{[X_1, X_3], [X_2, X_4]\} + \{[X_1, X_4], [X_2, X_3]\}, \end{aligned} \quad (253a)$$

$$\begin{aligned} [X_1, X_2, X_3, X_4, X_5, X_6] &= \frac{1}{96} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} \{[X_{\mu_1}, X_{\mu_2}, X_{\mu_3}, X_{\mu_4}], [X_{\mu_5}, X_{\mu_6}]\} \\ &= \frac{1}{32} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5, \mu_6} \{ \{[X_{\mu_1}, X_{\mu_2}], [X_{\mu_3}, X_{\mu_4}]\}, [X_{\mu_5}, X_{\mu_6}] \}. \end{aligned} \quad (253b)$$

In general,

$$\begin{aligned} [X_1, X_2, \dots, X_{2k}] &= \frac{1}{2^2(2k-2)!} \epsilon_{\mu_1 \dots \mu_{2k}} \{[X_{\mu_1}, X_{\mu_2}, \dots, X_{\mu_{2k-2}}], [X_{\mu_{2k-1}}, X_{\mu_{2k}}]\} \\ &= \frac{1}{2^4(2k-4)!} \epsilon_{\mu_1 \dots \mu_{2k}} \{ \{ \{ [X_{\mu_1}, X_{\mu_2}, \dots, X_{\mu_{2k-4}}], [X_{2k-3}, X_{2k-2}] \}, [X_{\mu_{2k-1}}, X_{\mu_{2k}}] \} \} \\ &= \frac{1}{2^6(2k-6)!} \epsilon_{\mu_1 \dots \mu_{2k}} \{ \{ \{ \{ [X_{\mu_1}, X_{\mu_2}, \dots, X_{\mu_{2k-6}}], [X_{\mu_{2k-5}}, X_{\mu_{2k-4}}] \}, [X_{\mu_{2k-3}}, X_{\mu_{2k-2}}] \}, [X_{\mu_{2k-1}}, X_{\mu_{2k}}] \} \} \\ &= \dots \\ &= \frac{1}{2^{2k-1}} \epsilon_{\mu_1 \dots \mu_{2k}} \{ \dots \{ [X_{\mu_1}, X_{\mu_2}], [X_{\mu_3}, X_{\mu_4}] \}, \dots, \}, [X_{\mu_{2k-5}}, X_{\mu_{2k-4}}], [X_{\mu_{2k-3}}, X_{\mu_{2k-2}}], [X_{\mu_{2k-1}}, X_{\mu_{2k}}] \}. \end{aligned} \quad (254)$$

In covariant form, (254) can be expressed as

$$\begin{aligned} [X_{a_1}, X_{a_2}, \dots, X_{a_{2k}}] &= \frac{1}{2^2(2k-2)!} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} \dots b_{a_{2k}} a_{2k+1}} \{[X_{b_{a_1}}, X_{b_{a_2}}, \dots, X_{b_{a_{2k-2}}}], [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \} \\ &= \frac{1}{2^4(2k-4)!} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} \dots b_{a_{2k}} a_{2k+1}} \{ \{ [X_{b_{a_1}}, X_{b_{a_2}}, \dots, X_{b_{a_{2k-4}}}], [X_{b_{a_{2k-3}}}, X_{b_{a_{2k-2}}}] \}, [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \} \\ &= \frac{1}{2^6(2k-6)!} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} \dots b_{a_{2k}} a_{2k+1}} \\ &\quad \times \{ \{ \{ \{ [X_{b_{a_1}}, X_{b_{a_2}}, \dots, X_{b_{a_{2k-6}}}], [X_{b_{a_{2k-5}}}, X_{b_{a_{2k-4}}}] \}, [X_{b_{a_{2k-3}}}, X_{b_{a_{2k-2}}}] \}, [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \} \} \\ &= \dots \\ &= \frac{1}{2^{2k-1}} \epsilon_{a_1 a_2 \dots a_{2k+1}} \epsilon_{b_{a_1} \dots b_{a_{2k}} a_{2k+1}} \\ &\quad \times \{ \dots \{ [X_{b_{a_1}}, X_{b_{a_2}}], [X_{b_{a_3}}, X_{b_{a_4}}] \}, \dots, \}, [X_{b_{a_{2k-5}}}, X_{b_{a_{2k-4}}}], [X_{b_{a_{2k-3}}}, X_{b_{a_{2k-2}}}], [X_{b_{a_{2k-1}}}, X_{b_{a_{2k}}}] \}. \end{aligned} \quad (255)$$

One may find that there exists a dimensional hierarchy:

$$2k \rightarrow 2k-2 \rightarrow 2k-4 \rightarrow \dots 4 \rightarrow 2, \quad (256)$$

and the non-commutativity of $2k$ -bracket is boiled down to its “constituent” algebra. Typically, when

$$[X_1, X_2] = [X_3, X_4] = \dots = [X_{2k-1}, X_{2k}] = i\ell^2, \quad (257)$$

the quantum Nambu geometry becomes a simple product of the two brackets:

$$[X_1, X_2, \dots, X_{2k-1}, X_{2k}] = (i\ell^2)^k = i^k \ell^{2k}. \quad (258)$$

C Winding number for $S^{2k-1} \rightarrow SO(2k)$ and tensor monopole charge

The non-trivial bundle topology of the $SO(2k)$ non-abelian monopole on S^{2k} is represented by the homotopy:

$$\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z}. \quad (259)$$

The corresponding Chern number is given by

$$c_k = \frac{1}{\mathcal{N}} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1}, \quad (260)$$

where g denotes the transition function on S^{2k-1} which takes the value in $SO(2k)$ group element and \mathcal{N} is a normalization constant defined so as to give $c_k = 1$ for the isomorphic map from S^{2k-1} to $SO(2k)$. The isomorphic map is given by

$$g = x_{2k} + i \sum_{i=1}^{2k-1} \gamma_i x_i, \quad (261)$$

where $(x_i, x_{2k}) \in S^{2k-1}$ are subject to $\sum_{i=1}^{2k-1} x_i x_i + x_{2k} x_{2k} = 1$ and γ_i ($i = 1, 2, \dots, 2k-1$) are the $SO(2k-1)$ gamma matrices. Obviously, $g^\dagger g = 1$. Around the north-pole $x_{2k} \simeq 1$ and $x_i \simeq 0$ ($i = 1, 2, \dots, 2k-1$), the transition function behaves as

$$g^\dagger \simeq 1, \quad dg \simeq i \sum_{\mu=1}^{2k-1} \gamma_\mu dx_\mu, \quad (262)$$

and the normalization constant \mathcal{N} is evaluated as

$$\begin{aligned} \mathcal{N} &= \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1} \\ &\sim \int_{S^{2k-1}} \text{tr}(\gamma_i dx_i)^{2k-1} = \int_{S^{2k-1}} dx_{i_1} dx_{i_2} \cdots dx_{i_{2k-1}} \text{tr}(\gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{2k-1}}) \\ &= (i)^{k-1} 2^{k-1} (2k-1)! \mathcal{A}(S^{2k-1}), \end{aligned} \quad (263)$$

where we used

$$\begin{aligned} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{2k-1}} &= (i)^{k-1} \epsilon_{i_1 i_2 \cdots i_{2k-1}} \mathbf{1}_{2^{k-1}} \\ dx_{i_1} dx_{i_2} \cdots dx_{i_{2k-1}} &= \epsilon_{i_1 i_2 \cdots i_{2k-1}} d^{2k-1}x. \end{aligned} \quad (264)$$

Consequently, the k th Chren number can be expressed as

$$c_k = \frac{(-i)^{k-1}}{(2k-1)! 2^{k-1} \mathcal{A}(S^{2k-1})} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1} = (-i)^{k-1} \frac{1}{(2\pi)^k} \frac{(k-1)!}{(2k-1)!} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1}. \quad (265)$$

In low dimensions, we have

$$\begin{aligned}
c_1 &= \frac{1}{2\pi} \int_{S^1} \text{tr}(-ig^\dagger dg), \\
c_2 &= -i \frac{1}{24\pi^2} \int_{S^3} \text{tr}(-ig^\dagger dg)^3, \\
c_3 &= -\frac{1}{480\pi^3} \int_{S^5} \text{tr}(-ig^\dagger dg)^5, \\
c_4 &= i \frac{1}{13440\pi^4} \int_{S^7} \text{tr}(-ig^\dagger dg)^7.
\end{aligned} \tag{266}$$

From the general integral expressio of c_k :

$$c_k = \int_{S^{2k-1}} \rho_{2k-1}, \tag{267}$$

we define

$$\rho_{2k-1} = (-i)^{k-1} \frac{1}{(2\pi)^k} \frac{(k-1)!}{(2k-1)!} \text{tr}(-ig^\dagger dg)^{2k-1}, \tag{268}$$

which satisfies

$$d\rho_{2k-1} = 0, \tag{269}$$

since $d[\text{tr}(-ig^\dagger dg)^{2k-1}] = -\text{tr}(-ig^\dagger dg)^{2k} = 0$. Due to the Poincaré lemma, ρ_{2k-1} is locally expressed as

$$\rho_{2k-1} = d\Lambda_{2k-2}. \tag{270}$$

Λ_{2k-2} corresponds to the $U(1)$ transition function of the $(2k-1)$ form gauge field [see (155)]. The associated $U(1)$ topological charge q_k is given by

$$q_k \equiv \int_{S^{2k-1}} d\Lambda_{2k-2} = \int_{S^{2k-1}} \rho_{2k-1} = c_k, \tag{271}$$

which is exactly equal to the k th Chern number and consistent with (143).

We also demonstrate that the pure gauge Chern-Simons action reproduces ρ_{2k-1} on the equator S^{2k-1} . The $SO(2k)$ non-abelian gauge fields on north and the south hemispheres are related as

$$A' = g^\dagger A g - ig^\dagger dg, \tag{272}$$

where g is given by

$$g = \frac{1}{\sqrt{1-x_{2k+1}^2}} (x_{2k} + i\gamma_i x_i). \tag{273}$$

Here, we used

$$\begin{aligned}
A &= i \frac{1}{2} (1 - x_{2k+1}) dg g^\dagger, \\
A' &= -i \frac{1}{2} (1 + x_{2k+1}) g^\dagger dg.
\end{aligned} \tag{274}$$

On the equator of S^{2k} , the transition function is reduced to (261):

$$g \xrightarrow{x_{2k+1}=0} x_{2k} + ix_i \gamma_i. \quad (275)$$

In the pure gauge

$$A = -ig^\dagger dg, \quad (276)$$

the Chern-Simons action (147) becomes to

$$L_{CS}^{2k-1} = (-i)^{k-1} \frac{k!(k-1)!}{(2k-1)!} \text{tr}(-ig^\dagger dg)^{2k-1}, \quad (277)$$

where we used

$$s(k) = k \int_0^1 dt (t - t^2)^{k-1} = \frac{k!(k-1)!}{(2k-1)!}. \quad (278)$$

On the equator S^{2k-1} , (277) is reduced to (268) up to a proportional factor:

$$L_{CS}^{2k-1} = i^{2k-1} (2\pi)^k k! \rho_{2k-1}. \quad (279)$$

D Linking number between membranes

The description here is mainly based on [86, 76, 85]. The tensor Chern-Simons action is given by

$$S = -\frac{2}{(2k-1)!} \int d^{4k-1}x J_{\mu_1 \mu_2 \dots \mu_{2k-1}} C^{\mu_1 \mu_2 \dots \mu_{2k-1}} \\ + \frac{1}{\theta} \frac{1}{(2k-1)!(2k)!} \int d^{4k-1}x \epsilon^{\mu_1 \mu_2 \dots \mu_{4k-1}} C_{\mu_1 \mu_2 \dots \mu_{2k-1}} G_{\mu_{2k} \mu_{2k+1} \dots \mu_{4k-1}}. \quad (280)$$

In accordance with the Chern-Simons effective field theory for A-class topological insulator (216), θ should take

$$\theta = 2\pi m^k, \quad (281)$$

however in the following we render θ to an arbitrary parameter. We derive the Hopf Lagrangian by integrating out the Chern-Simons gauge field. The equation for the Chern-Simons field is derived as

$$J_{\mu_1 \mu_2 \dots \mu_{2k-1}} = \frac{1}{\theta(2k)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{4k-1}} G^{\mu_{2k} \mu_{2k+1} \dots \mu_{4k-1}}, \quad (282)$$

or

$$G^{\mu_1 \mu_2 \dots \mu_{2k}} = -\theta \frac{1}{(2k-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{4k-1}} J_{\mu_{2k+1} \mu_{2k+2} \dots \mu_{4k-1}}. \quad (283)$$

It is obvious that the current satisfies a generalized current conservation law:

$$\partial^{\mu_i} J_{\mu_1 \dots \mu_i \dots \mu_{2k-1}} = 0 \quad (i = 1, 2, \dots, 2k-1). \quad (284)$$

In a Coulomb like gauge $\partial^\mu C_{\mu_1 \dots \mu_{2k-1}} = 0$, the Chern-Simons field is expressed as

$$C^{\mu_1 \mu_2 \dots \mu_{2k-1}} = -\theta \frac{1}{(2k-1)!} \epsilon^{\mu_1 \mu_2 \dots \mu_{4k-1}} \partial_{\mu_{2k}} \frac{1}{\partial^2} J_{\mu_{2k+1} \dots \mu_{4k-1}}, \quad (285)$$

where we used the formula

$$\frac{1}{\epsilon \partial} = -\frac{1}{(2k-1)!} \epsilon \partial \frac{1}{\partial^2}. \quad (286)$$

By substituting (285) to the interaction and the Chern-Simons action of (280), we have

$$S_{Hopf} = \theta \frac{1}{((2k-1)!)^2} \int d^{4k-1}x \epsilon^{\mu_1 \mu_2 \dots \mu_{4k-1}} J_{\mu_1 \mu_2 \dots \mu_{2k-1}} \partial_{\mu_{2k}} \frac{1}{\partial^2} J_{\mu_{2k+1} \dots \mu_{4k-1}}. \quad (287)$$

In the thin membrane limit³⁰, (287) yields the linking number of two $(2k-2)$ branes:

$$S_{Hopf} \Rightarrow \theta L(V_1, V_2). \quad (290)$$

Here $L(V_1, V_2)$ denotes the higher dimensional generalization of the linking number:

$$L(V_1, V_2) = \frac{1}{((2k-1)!)^2 \mathcal{A}(S^{4k-2})} \oint_{V_1} dx^{\mu_1 \mu_2 \dots \mu_{2k-1}} \oint_{V_2} dx'^{\mu_{2k+1} \mu_{2k+2} \dots \mu_{4k-1}} \epsilon_{\mu_1 \mu_2 \dots \mu_{4k-1}} \frac{x_{\mu_{2k}}(\sigma) - x'_{\mu_{2k}}(\sigma')}{|x(\sigma) - x'(\sigma')|^{4k-1}}, \quad (291)$$

with

$$\begin{aligned} dx^{\mu_1 \mu_2 \dots \mu_{2k-1}} &\equiv d\sigma^0 d\sigma^1 \dots d\sigma^{2k-2} \frac{\partial(x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_{2k-1}})}{\partial(\sigma^0, \sigma^1, \dots, \sigma^{2k-2})}, \\ dy^{\mu_1 \mu_2 \dots \mu_{2k-1}} &\equiv d\sigma'^0 d\sigma'^1 \dots d\sigma'^{2k-2} \frac{\partial(y^{\mu_1}, y^{\mu_2}, \dots, y^{\mu_{2k-1}})}{\partial(\sigma'^0, \sigma'^1, \dots, \sigma'^{2k-2})}. \end{aligned} \quad (292)$$

With use of the normalized relative coordinates

$$z_\mu(\sigma, \sigma') \equiv \frac{x_\mu(\sigma) - x'_\mu(\sigma')}{|x(\sigma) - x'(\sigma')|}, \quad (293)$$

the the linking number (291) is concisely expressed as

$$L(V_1, V_2) = \frac{1}{(4k-2)! \mathcal{A}(S^{4k-2})} \int dz^{\mu_1} dz^{\mu_2} \dots dz^{\mu_{4k-2}} \epsilon_{\mu_1 \mu_2 \dots \mu_{4k-1}} z_{\mu_{4k-1}}, \quad (294)$$

where

$$dz^{\mu_1} dz^{\mu_2} \dots dz^{\mu_{4k-2}} \equiv d\sigma_0 d\sigma_1 \dots d\sigma_{2k-2} d\sigma'_0 d\sigma'_1 \dots d\sigma'_{2k-2} \frac{\partial(z^{\mu_1}, z^{\mu_2}, \dots, z^{\mu_{4k-2}})}{\partial(\sigma_0, \dots, \sigma_{2k-2}, \sigma'_0, \dots, \sigma'_{2k-2})}. \quad (295)$$

Here, we used the formula of the determinant (297)³¹. It should be noticed that the integral in (294)

$$\mathcal{A}(S^{4k-2}) = \frac{1}{(4k-2)!} \int dz^{\mu_1} dz^{\mu_2} \dots dz^{\mu_{4k-2}} \epsilon_{\mu_1 \mu_2 \dots \mu_{4k-1}} z_{\mu_{4k-1}} \quad (298)$$

³⁰The thin membrane current is given by

$$J^{\mu_1 \mu_2 \dots \mu_{2k-1}}(x) = \int d^{2k-1}\sigma \frac{\partial(y^{\mu_1}, y^{\mu_2}, \dots, y^{\mu_{2k-1}})}{\partial(\sigma^0, \sigma^1, \dots, \sigma^{2k-2})} \delta^{(4k-1)}(x - y(\sigma)), \quad (288)$$

where

$$\frac{\partial(y^{\mu_1}, y^{\mu_2}, \dots, y^{\mu_{p+1}})}{\partial(\sigma_0, \sigma_1, \dots, \sigma_p)} \equiv \epsilon_{\alpha_1 \alpha_2 \dots \alpha_{p+1}} \frac{\partial y^{\mu_1}}{\partial \sigma_{\alpha_1}} \frac{\partial y^{\mu_2}}{\partial \sigma_{\alpha_2}} \dots \frac{\partial y^{\mu_{p+1}}}{\partial \sigma_{\alpha_{p+1}}} \quad (289)$$

denotes the Jacobian.

³¹ In the map from a set of $2k$ coordinates, $\sigma_1, \sigma_2, \dots, \sigma_{2k}$, to another set of $2k$ coordinates, $z_\mu = z_\mu(\sigma_1, \sigma_2, \dots, \sigma_{2k})$ ($\mu = 1, 2, \dots, 2k$), the volume density is given by

$$D \equiv \frac{\partial(z_1, z_2, \dots, z_{2k})}{\partial(\sigma_1, \sigma_2, \dots, \sigma_{2k})} \equiv \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} \frac{\partial z_{\mu_1}}{\partial \sigma_1} \frac{\partial z_{\mu_2}}{\partial \sigma_2} \dots \frac{\partial z_{\mu_{2k}}}{\partial \sigma_{2k}}, \quad (296)$$

represents the area of S^{4k-2} whose coordinates are z_μ ($\sum_{\mu=1}^{4k-1} z_\mu z_\mu = 1$). Thus, the linking number can be alternatively understood as the winding number from the world-volumes of two $(2k-2)$ membranes to S^{4k-2} :

$$(S^{2k-2} \times S^1) \times (S^{2k-2} \times S^1) \rightarrow S^{4k-2}. \quad (299)$$

For $k=1$, (291) is reduced to the original Gauss linking [102, 103, 104]:

$$L(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx^\mu \oint_{C_2} dx'^\rho \epsilon_{\mu\nu\rho} \frac{x_\nu(\sigma) - x'_\nu(\sigma')}{|x(\sigma) - x'(\sigma')|^3}, \quad (300)$$

and similarly (299) becomes to

$$T^2 \equiv S^1 \times S^1 \rightarrow S^2. \quad (301)$$

References

- [1] C.L. Kane, E.J. Mele, “*Z₂ Topological Order and the Quantum Spin Hall Effect*”, Phys. Rev. Lett. 95 (2005) 146802; cond-mat/0506581.
- [2] B. Andrei Bernevig, Shou-Cheng Zhang, “*Quantum Spin Hall Effect*”, Phys. Rev. Lett. 96 (2006) 106802, cond-mat/0504147.
- [3] B. Andrei Bernevig, Taylor L. Hughes, Shou-Cheng Zhang, “*Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells*”, Science, 314 (2006) 1757; cond-mat/0611399.
- [4] Markus Koenig et al, “*Quantum Spin Hall Insulator State in HgTe Quantum Wells*”, Science, 318 (2007) 766; arXiv:0710.0582.
- [5] X.-L. Qi and S.-C. Zhang, “*The quantum spin Hall effect and topological insulators*”, Physics Today 63 (2010) 33-38; arXiv:1001.1602.
- [6] M.Z. Hasan and C.L. Kane, “*Topological insulators*”, Rev. Mod. Phys. 82 (2010) 3045.
- [7] X.-L. Qi and S.-C. Zhang, “*Topological insulators and superconductors*”, Rev. Mod. Phys. 83 (2011) 1057.
- [8] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, Andreas W. W. Ludwig, “*Classification of topological insulators and superconductors in three spatial dimensions*”, Phys. Rev. B 78 (2008) 195125; arXiv:0803.2786.
- [9] A. Kitaev, “*Periodic table for topological insulators and superconductors*”, Proceedings of the L.D.Landau Memorial Conference Advances in Theoretical Physics, June 22-26 (2008); arXiv:0901.2686.

which can be rewritten as

$$D = \frac{1}{(2k)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} \frac{\partial(z_{\mu_1}, z_{\mu_2}, \dots, z_{\mu_{2k}})}{\partial(\sigma_1, \sigma_2, \dots, \sigma_{2k})} = \frac{1}{(k!)^2} \epsilon_{\mu_1 \mu_2 \dots \mu_{2k}} \frac{\partial(z_{\mu_1}, z_{\mu_2}, \dots, z_{\mu_k})}{\partial(\sigma_1, \sigma_2, \dots, \sigma_k)} \frac{\partial(z_{\mu_{k+1}}, z_{\mu_{k+2}}, \dots, z_{\mu_{2k}})}{\partial(\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{2k})}. \quad (297)$$

- [10] Shinsei Ryu, Andreas Schnyder, Akira Furusaki, Andreas Ludwig “*Topological insulators and superconductors: ten-fold way and dimensional hierarchy*”, New J. Phys. 12 (2010) 065010; arXiv:0912.2157.
- [11] X-L. Qi, T. Hughes, S-C. Zhang, “*Topological Field Theory of TR Invariant Insulators*”, Phys. Rev. B 78 (2008) 195424-43; arXiv:0802.3537.
- [12] Titus Neupert, Luiz Santos, Shinsei Ryu, Claudio Chamon, and Christopher Mudry, “*Fractional topological liquids with time-reversal symmetry and their lattice realization*”, Phys. Rev. B 84 (2011) 165107; arXiv:1106.3989.
- [13] Luiz Santos, Titus Neupert, Shinsei Ryu, Claudio Chamon, and Christopher Mudry, “*Time-reversal symmetric hierarchy of fractional incompressible liquids*”, Phys. Rev. B 84 (2011) 165138; arXiv:1108.2440.
- [14] B. Estienne, N. Regnault, B. A. Bernevig, “*D-Algebra Structure of Topological Insulators*”, Phys. Rev. B 86 (2012) 241104(R); arXiv:1202.5543.
- [15] Titus Neupert, Luiz Santos, Shinsei Ryu, Claudio Chamon, Christopher Mudry, “*Noncommutative geometry for three-dimensional topological insulators*”, Phys. Rev. B 86 (2012) 035125; arXiv:1202.5188.
- [16] S.M. Girvin and T. Jach, “*Formalism for the quantum Hall effect: Hilbert space of analytic functions*”, Phys.Rev. B29 (1984) 5617.
- [17] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, “*Collective-Excitation Gap in the Fractional Quantum Hall Effect*”, Phys. Rev. Lett. 54 (1985) 581; “*Magneto-roton theory of collective excitations in the fractional quantum Hall effect*”, Phys. Rev. B33 (1986) 2481.
- [18] Z.F. Ezawa, G. Tsitsishvili, K. Hasebe, “*Noncommutative Geometry, Extended W(infty) Algebra and Grassmannian Solitons in Multicomponent Quantum Hall Systems*”, Phys.Rev. B67 (2003) 125314; hep-th/0209198.
- [19] F. D. M. Haldane, “*Geometrical Description of the Fractional Quantum Hall Effect*”, Phys.Rev. Lett. 107 (2011) 116801; arXiv:1106.3375.
- [20] Yoichiro Nambu, “*Generalized Hamiltonian Dynamics*”, Phys.Rev.D7 (1973) 2405-2412.
- [21] Thomas Curtright, Cosmas Zachos, “*Classical and Quantum Nambu Mechanics*”, Phys.Rev.D68 (2003) 085001; hep-th/0212267.
- [22] M. M. Sheikh-Jabbari, “*Tiny Graviton Matrix Theory: DLCQ of IIB Plane-Wave String Theory, A Conjecture*”, JHEP 0409 (2004) 017; hep-th/0406214.
- [23] M. M. Sheikh-Jabbari, M. Torabian, “*Classification of All 1/2 BPS Solutions of the Tiny Graviton Matrix Theory*”, JHEP 0504 (2005) 001; hep-th/0501001.

- [24] Joshua DeBellis, Christian Saemann, Richard J. Szabo, “*Quantized Nambu-Poisson Manifolds and n -Lie Algebras*”, J.Math.Phys.51 (2010) 122303; arXiv:1001.3275.
- [25] Yi Li, Congjun Wu, “*High-Dimensional Topological Insulators with Quaternionic Analytic Landau Levels*”, Phys. Rev. Lett. 110 (2013) 216802; arXiv:1103.5422.
- [26] Yi Li, Shou-Cheng Zhang, Congjun Wu, “*Topological insulators with $SU(2)$ Landau levels*”, Phys. Rev. Lett. 111 (2013) 186803; arXiv:1208.1562.
- [27] S. C. Zhang and J. P. Hu, “*A Four Dimensional Generalization of the Quantum Hall Effect*”, Science 294 (2001) 823; cond-mat/0110572.
- [28] F.D.M. Haldane, “*Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states*”, Phys. Rev. Lett. 51 (1983) 605-608.
- [29] D. Karabali, V. P. Nair, “*Quantum Hall Effect in Higher Dimensions*”, Nucl.Phys. B641 (2002) 533; hep-th/0203264.
- [30] B. A. Bernevig, J. P. Hu, N. Toumbas, S. C. Zhang, “*The eight dimensional quantum Hall effect and the octonions*”, Phys.Rev.Lett. 91 (2003) 236803; cond-mat/0306045.
- [31] K. Hasebe and Y. Kimura, “*Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres*”, Phys.Lett. B 602 (2004) 255; hep-th/0310274.
- [32] Kazuki Hasebe, ‘*Hopf Maps, Lowest Landau Level, and Fuzzy Spheres*’, SIGMA 6 (2010) 071.
- [33] Norman Steenrod, Section 18 in “*The Topology of Fibre Bundles*”, Princeton University Press, (1951).
- [34] R.C. Myers, “*Dielectric-Branes*”, JHEP 9912 (1999) 022; hep-th/9910053.
- [35] Yusuke Kimura, “*Nonabelian gauge field and dual description of fuzzy sphere*”, JHEP 0404 (2004) 058; hep-th/0402044.
- [36] K. Hasebe, Y. Kimura, “*Fuzzy Supersphere and Supermonopole*”, Nucl.Phys. B709 (2005) 94-114; hep-th/0409230.
- [37] Kazuki Hasebe, “*Supersymmetric Quantum Hall Effect on a Fuzzy Supersphere*”, Phys.Rev.Lett. 94 (2005) 206802; hep-th/0411137.
- [38] Dana Mihai, George Sparling, Philip Tillman, “*Non-Commutative Time, the Quantum Hall Effect and Twistor Theory*”, cond-mat/0401224.
- [39] K. Hasebe, “*Split-Quaternionic Hopf Map, Quantum Hall Effect and Twistor Theory*”, Phys.Rev.D81 (2010) 041702; arXiv:0902.2523.
- [40] R.I. Nepomechie, “*Magnetic monopoles from antisymmetric tensor gauge fields*”, Phys. Rev. D 31 (1985) 1921.

- [41] Claudio Teitelboim, “*Gauge invariance for extended objects*”, Phys.Lett. B 167 (1986) 63-68, “*Monopoles of higher rank*”, Phys.Lett. B 167 (1986) 69-72.
- [42] Michael Kalb, P. Ramond, “*Classical direct interstring action*”, Phys.Rev. D 9 (1974) 2273-2284.
- [43] S.C. Zhang, T.H. Hansson, S. Kivelson “*An Effective Field Theory Model For The Fractional Quantum Hall Effect*”, Phys.Rev.Lett.62 (1988) 82-85.
- [44] S.C. Zhang, “*The Chern-Simons-Landau-Ginzburg Theory Of The Fractional Quantum Hall Effect*”, Int.J.Mod.Phys. B6 (1992) 25-58.
- [45] S. M. Girvin, A. H. MacDonald “*Off-diagonal long-range order, oblique confinement, and the fractional quantum Hall effect*”, Phys. Rev. Lett. 58 (1987) 1252-1255.
- [46] N. Read, “*Order Parameter and Ginzburg-Landau Theory for the Fractional Quantum Hall Effect*”, Phys. Rev. Lett. 62 (1989) 86.
- [47] Shinsei Ryu, Tadashi Takayanagi, “*Topological Insulators and Superconductors from D-branes*”, Phys. Lett. B 693 (2010) 175-179; arXiv:1001.0763. “*Topological Insulators and Superconductors from String Theory*”, Phys.Rev.D82 (2010) 086014; arXiv:1007.4234.
- [48] Akira Furusaki, Naoto Nagaosa, Kentaro Nomura, Shinsei Ryu, Tadashi Takayanagi, “*Topological Insulators and Superconductors from String Theory*”, Comptes Rendus Physique 14 (2013) 871-883; arXiv:1211.0533.
- [49] E. Bergshoeff, D. S. Berman, J. P. van der Schaar, P. Sundell, “*A Noncommutative M-Theory Five-brane*”, Nucl.Phys. B590 (2000) 173-197; hep-th/0005026.
- [50] Shoichi Kawamoto, Naoki Sasakura, “*Open membranes in a constant C-field background and noncommutative boundary strings*”, JHEP 0007 (2000) 014; hep-th/0005123.
- [51] Chong-Sun Chu, Gurdeep S. Sehmbi, “*D1-strings in large RR 3-form flux, quantum Nambu geometry and M5-branes in the C-field*”, J. Phys.A 45 (2012) 055401; arXiv:1110.2687.
- [52] F.A. Berezin, “*General Concept of Quantization*”, Commun.Math. Phys. 40 (1975) 153-174.
- [53] J. Hoppe, “*Quantum Theory of a Massless Relativistic Surface and a Two-dimensional Bound State Problem*”, MIT PhD Thesis (1982).
- [54] J. Madore, “*The Fuzzy Sphere*”, Class. Quant. Grav. 9 (1992) 69.
- [55] H. Grosse, C. Klimcik, P. Presnajder, “*On Finite 4D Quantum Field Theory in Non-Commutative Geometry*”, Commun.Math.Phys. 180 (1996) 429-438; hep-th/9602115.
- [56] D. Kabat, W. Taylor, “*Spherical membranes in Matrix theory*”, Adv.Theor.Math.Phys.2 (1998) 181-206; hep-th/9711078.

- [57] S. Ramgoolam, “*On spherical harmonics for fuzzy spheres in diverse dimensions*”, Nucl.Phys. B610 (2001) 461-488; hep-th/0105006.
- [58] P. M. Ho and S. Ramgoolam, “*Higher dimensional geometries from matrix brane constructions*”, Nucl.Phys.B 627 (2002) 266; hep-th/0111278.
- [59] Yusuke Kimura, “*Noncommutative Gauge Theory on Fuzzy Four-Sphere and Matrix Model*”, Nucl.Phys. B637 (2002) 177-198 ; hep-th/0204256.
- [60] Yusuke Kimura, “*On Higher Dimensional Fuzzy Spherical Branes*”, Nucl.Phys. B664 (2003) 512-530; hep-th/0301055.
- [61] Paul Dirac, “*Quantised Singularities in the Electromagnetic Field*”, Proc. Roy. Soc. (London) A 133 (1931) 60.
- [62] Rafael D. Sorkin, “*Kaluza-Klein Monopole*”, Phys.Rev.Lett. 51 (1983) 87-90.
- [63] David J. Gross, Malcolm J. Perry, “*Magnetic Monopoles in Kaluza-Klein Theories*”, Nuclear Physics B 226 (1983) 29-48.
- [64] Malcolm J. Perry, “*Non-Abelian Kaluza-Klein Monopoles*”, Phys. Lett. B 137 (1983) 171-174.
- [65] Chen Ning Yang, “*Generalization of Dirac’s monopole to SU_2 gauge fields*”, J. Math. Phys. 19 (1978) 320.
- [66] Z. Horvath and L. Palla, “*Spontaneous compactification and monopoles in higher dimensions*”, Nucl.Phys. B142 (1978) 327.
- [67] D. H. Tchrakian, “ *N -dimensional instantons and monopoles*”, J.Math. Phys. 21 (1980) 166.
- [68] Bernard Grossman, Thomas W. Kephart, James D. Stasheff, “*Solutions To Yang-Mills Field Equations In Eight-Dimensions And The Last Hopf Map*”, Commun.Math.Phys. 96 (1984) 431, Erratum-ibid.100 (1985) 311.
- [69] C. Sacloglu, “*Scale invariant gauge theories and self-duality in higher dimensions*”, Nucl.Phys. B277 (1986) 487.
- [70] Brian P. Dolan, “*The Spectrum of the Dirac Operator on Coset Spaces with Homogeneous Gauge Fields*”, JHEP 0305 (2003) 018; hep-th/0302122.
- [71] A. Lichnerowicz, “*Spineurs harmonique*”, C. R. Acad. Sci. Paris Sér. A 257 (1963)
- [72] S. Sulanke, “*Berechnung des Spektrums des Quadrates des Dirac-Operators auf der Sphäre*”, Ph. D. thesis, Humboldt-UniversitNat, Berlin 1978.
- [73] A. Trautman, “*The Dirac operator on hypersurfaces*”, Acta Phys.Pol.B26 (1995) 1283-1310; hep-th/9810018.

- [74] R. Camporesi, A. Higuchi, “*On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces*”, J.Geom.Phys. 20 (1996) 1-18; gr-qc/9505009.
- [75] A.P. Balachandran, Giorgio Immirzi, Joochan Lee, Peter Presnajder, “*Dirac Operators on Coset Spaces*”, J.Math.Phys. 44 (2003) 4713-4735; hep-th/0210297.
- [76] Y.S. Wu, A. Zee, “*Membranes, higher Hopf maps, and phase interactions*”, Phys. Lett. B 207 (1988) 39.
- [77] Eugene Demler, Shou-Cheng Zhang, “*Non-Abelian Holonomy of BCS and SDW Quasiparticles*”, Annals Phys. 271 (1999) 83-119; cond-mat/9805404.
- [78] Luis Alvarez-Gaumé and Paul Ginsparg, “*The Topological Meaning of Non-Abelian Anomalies*”, Nucl.Phys. B 243 (1984) 449, “*The Structure of Gauge and Gravitational Anomalies*”, Ann. Phys. NY 161 (1985) 423.
- [79] D. Arovas, J. R. Schrieffer, F. Wilczek, “*Fractional Statistics and the Quantum Hall Effect*”, Phys.Rev. Lett. 53 (1984) 722.
- [80] B. I. Halperin, “*Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States*”, Phys. Rev. Lett. 52 (1984) 1583-1586.
- [81] D. Ravenel, A. Zee, “*Spin and statistics of solitons and J-Homomorphism*”, Commun. Math. Phys. 98 (1985) 239.
- [82] F. Wilczek, A. Zee, “*Linking numbers, spin, and statistics of solitons*”, Phys. Rev. Lett. 51 (1983) 2250-2252.
- [83] Mark J. Bowick, Dimitra Karabali, L.C.R. Wijewardhana, “*Fractional spin via canonical quantization of the $O(3)$ nonlinear sigma model*”, Nucl. Phys. B 271 (1986) 417-428.
- [84] R. Bott, L. W. Tu, “*Differential Forms in Algebraic Topology*”, Springer, p. 293.
- [85] R.I. Nepomechie and A. Zee, “*Phase interaction between extended objects*”, I.A. Batalin et al (Eds.) Quantum Field Theory and Quantum Statistics, 2 (1984) 467-472.
- [86] C-H Tze, S. Nam, “*Topological phase entanglements of membrane solitons in division algebra sigma models with Hopf term*”, Annals of Phys. 193 (1989) 419.
- [87] Gary T. Horowitz, Mark Srednicki, “*A quantum field theoretic description of linking numbers and their generalization*”, Comm. Math. Phys. 130 (1990) 83-94.
- [88] B. Andrei Bernevig, Chyh-Hong Chern, Jiang-Ping Hu, Nicolaos Toumbas, Shou-Cheng Zhang, “*Effective field theory description of the higher dimensional quantum Hall liquid*”, Ann. Phys. 300 (2002) 185; cond-mat/0206164.
- [89] F. Wilczek, “*Magnetic Flux, Angular Momentum, and Statistics*”, Phys. Rev. Lett. 48 (1982) 1144-1146.

- [90] F. Wilczek, “*Quantum Mechanics of Fractional-Spin Particle*”, Phys. Rev. Lett. 49 (1982) 957-959.
- [91] Y. N. Kafiev, *Four-Dimensional σ -Models and Geometry of Yong-Mills Fields*, Nucl.Phys. B 178 (1981) 177.
- [92] T. Banks, W. Fisher, S. H. Shenker and L. Susskind, “*M theory as a matrix model: a conjecture*”, Phys.Rev.D 55 (1997) 5112; hep-th/9610043.
- [93] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, “*A large- N reduced model as superstring*”, Nucl.Phys.B 498 (1997) 467; hep-th/9705128.
- [94] P.K. Townsend, “*p-Brane Democracy*”, hep-th/9507048.
- [95] Anirban Basu, Jeffrey A. Harvey, “*The M2-M5 Brane System and a Generalized Nahm’s Equation*”, Nucl.Phys. B713 (2005) 136-150; hep-th/0412310.
- [96] Jonathan Bagger, Neil Lambert, “*Modeling Multiple M2’s*”, Phys.Rev.D75 (2007) 045020; hep-th/0611108, “*Gauge Symmetry and Supersymmetry of Multiple M2-Branes*”, Phys.Rev.D77 (2008) 065008; arXiv:0711.0955.
- [97] Andreas Gustavsson, “*Algebraic structures on parallel M2-branes*”, Nucl.Phys.B811 (2009) 66-76; arXiv:0709.1260.
- [98] Jonathan Bagger, Neil Lambert, Sunil Mukhi, Constantinos Papageorgakis, “*Multiple Membranes in M-theory*”, Phys. Rep. 527 (2013) 1-100; arXiv:1203.3546.
- [99] Dionysios Mylonas, Peter Schupp, Richard J. Szabo, “*Non-Geometric Fluxes, Quasi-Hopf Twist Deformations and Nonassociative Quantum Mechanics*”, arXiv:1312.1621.
- [100] Ioannis Bakas, Dieter Lust, “*3-Cocycles, Non-Associative Star-Products and the Magnetic Paradigm of R-Flux String Vacua*”, arXiv:1309.3172.
- [101] Shou-Cheng Zhang, “*To see a world in a grain of sand*”, A chapter for John Wheeler’s 90’s birthday festschrift; hep-th/0210162.
- [102] C. F. Gauss, “*Werke*”, Königliche Gesellschaft der Wissenschaften, Göttingen 1833, Vol. V, 605, note of Jan. 22.
- [103] J. C. Maxwell, “*A Treatise on Electricity and Magnetism*”, Oxford, Clarendon, England, 1873, 40-42.
- [104] Abhay Ashtekar, Alejandro Corichi, “*Gauss linking number and the electromagnetic uncertainty principle*”, Phys. Rev. D 56 (1997) 2073-2079.